

Math 233 - Homework 2
February 18, 2021

Name key Score _____

The following problems are from the suggested homework. Show all work to receive full credit. Supply explanations when necessary. This assignment is due on February 25.

1. (1 point) Let $\vec{r}(t) = 3e^t \hat{i} + 2e^{-3t} \hat{j} + 4e^{2t} \hat{k}$. Compute the unit tangent vector at $t = \ln 2$.

$$\vec{r}'(t) = 3e^t \hat{i} - 6e^{-3t} \hat{j} + 8e^{2t} \hat{k}$$

$$\begin{aligned}\vec{r}'(\ln 2) &= 3(\ln 2) \hat{i} - 6(2^{-3}) \hat{j} + 8(4) \hat{k} \\ &= 6\hat{i} - \frac{3}{4}\hat{j} + 32\hat{k}\end{aligned}$$

$$\|\vec{r}'(\ln 2)\| = \sqrt{36 + \frac{9}{16} + 1024} = \frac{\sqrt{16969}}{4}$$

$$\hat{T}(\ln 2) = \frac{24\hat{i} - 3\hat{j} + 128\hat{k}}{\sqrt{16969}}$$

$$\approx \langle 0.184, -0.023, 0.983 \rangle$$

2. (2 points) The acceleration function, initial velocity, and initial position of a particle are $\vec{a}(t) = -5 \cos t \hat{i} - 5 \sin t \hat{j}$, $\vec{v}(0) = 9\hat{i} + 2\hat{j}$, and $\vec{r}(0) = 5\hat{i}$. Find $\vec{v}(t)$ and $\vec{r}(t)$.

$$\vec{a}(t) = -5 \cos t \hat{i} - 5 \sin t \hat{j} \Rightarrow \vec{v}(t) = -5 \sin t \hat{i} + 5 \cos t \hat{j} + \vec{C}_1$$

$$\vec{v}(0) = 5\hat{j} + \vec{C}_1 = 9\hat{i} + 2\hat{j} \Rightarrow \vec{C}_1 = 9\hat{i} - 3\hat{j}$$

$$\begin{aligned}\vec{v}(t) &= (-5 \sin t + 9)\hat{i} + (5 \cos t - 3)\hat{j} \Rightarrow \vec{r}(t) = (5 \cos t + 9t)\hat{i} \\ &\quad + (5 \sin t - 3t)\hat{j} + \vec{C}_2\end{aligned}$$

$$\vec{r}(0) = 5\hat{i} + \vec{C}_2 = 5\hat{i} \Rightarrow \vec{C}_2 = \vec{0}$$

$$\vec{r}(t) = (5 \cos t + 9t)\hat{i} + (5 \sin t - 3t)\hat{j}$$

3. (1 point) Locate the highest point on the curve $\vec{r}(t) = (6t, 6t - t^2)$.

$$y(t) = 6t - t^2$$

$$\begin{aligned}y'(t) &= 6 - 2t = 0 \\ \Rightarrow t &= 3\end{aligned}$$

$$\begin{aligned}y''(t) &= -2 < 0 \quad (\text{CONCAVE DOWN}) \\ \Rightarrow t &= 3 \text{ gives} \\ &\quad \text{A MAX.}\end{aligned}$$

HIGHEST POINT IS AT

$$\vec{r}(3) = 18\hat{i} + 9\hat{j}$$

OR THE POINT

$$(18, 9)$$

Turn over.

4. (2 points) Find the arc length of the curve $\vec{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle$ over the interval $0 \leq t \leq \pi$.

$$\vec{r}'(t) = 2 \cos t \hat{i} + 5 \hat{j} - 2 \sin t \hat{k}$$

Arc Length =

$$\|\vec{r}'(t)\| = \sqrt{4 \cos^2 t + 25 + 4 \sin^2 t} \\ = \sqrt{29}$$

$$\int_0^\pi \sqrt{29} dt = \sqrt{29} t \Big|_0^\pi \\ = \boxed{\sqrt{29} \pi}$$

5. (2 points) Find the curvature of $\vec{r}(t) = \sqrt{2}t \hat{i} + e^t \hat{j} + e^{-t} \hat{k}$ at the point $P(0, 1, 1)$.

$$\vec{v}(t) = \sqrt{2} \hat{i} + e^t \hat{j} - e^{-t} \hat{k}$$

$$\vec{a}(t) = e^t \hat{j} + e^{-t} \hat{k}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} \\ = 2 \hat{i} - \sqrt{2} e^{-t} \hat{j} + \sqrt{2} e^t \hat{k}$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{4 + 2e^{-2t} + 2e^{2t}}$$

Point $P(0, 1, 1)$ corresponds to $t=0$.

$$K(t=0) = \frac{\|\vec{v}(0) \times \vec{a}(0)\|}{\|\vec{v}(0)\|^3}$$

$$= \frac{\sqrt{8}}{(\sqrt{2+1+1})^3} = \frac{\sqrt{8}}{(\sqrt{4})^3}$$

$$= \frac{\sqrt{8}}{\sqrt{64}} = \boxed{\frac{1}{\sqrt{8}}}$$

6. (2 points) A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 m. Find the minimum velocity necessary to achieve this range. (Use $g = 9.8 \text{ m/s}^2$.)

$$\vec{r}(t) = v_0 \cos 8^\circ t \hat{i} + (-4.9t^2 + v_0 \sin 8^\circ t) \hat{j}$$

$$v_0 \cos 8^\circ t = 50 \quad \text{when} \quad -4.9t^2 + v_0 \sin 8^\circ t = 0$$

$$t(-4.9t + v_0 \sin 8^\circ) = 0$$

$$t = \frac{v_0 \sin 8^\circ}{4.9}$$

$$v_0^2 = \frac{245}{\cos 8^\circ \sin 8^\circ} \approx 1777.698$$

$$\Rightarrow v_0 \approx 42.16 \text{ m/s}$$