Math 233 - Quiz 4

March 25, 2021

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. This quiz is due April 1.

1. (2 points) Use the definition of differentiability to show that $f(x,y) = 2x^2 - 10y$ is differentiable on $\mathbb{R}^2$.

$f_x(x,y) = 4x$, $f_y(x,y) = -10$

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = \left[2(x+\Delta x)^2 - 10(y+\Delta y)\right] - \left[2x^2 - 10y\right]$

$= \cancel{2x^2} + \underline{4x\Delta x} + \underline{\underline{2\Delta x^2}} - \cancel{10y} - \underline{\underline{10\Delta y}} - \cancel{2x^2} + \cancel{10y}$

$= 4x\,\Delta x - 10\,\Delta y + 2\Delta x\,\Delta x + 0\,\Delta y$

$= f_x\,\Delta x + f_y\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$, WHERE $\epsilon_1 = 2\Delta x$, $\epsilon_2 = 0$

Δz HAS THE CORRECT FORM, AND $\epsilon_1 \to 0$ AND $\epsilon_2 \to 0$ AS $(\Delta x, \Delta y) \to (0,0)$. TRUE ON ALL $\mathbb{R}^2$.

2. (3 points) Find equations of the tangent plane and the normal line (parametric and symmetric) at the point $(1,0,0)$.

$$z + 1 = xe^y \cos z$$

$F(x,y,z) = xe^y \cos z - z - 1$

THE GIVEN SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$

$\vec{\nabla} F(x,y,z) = e^y \cos z\,\hat{i} + xe^y \cos z\,\hat{j} + (-xe^y \sin z - 1)\,\hat{k}$

$\vec{n} = \vec{\nabla} F(1,0,0) = \hat{i} + \hat{j} - \hat{k}$

TAN PLANE:

$1(x-1) + 1(y-0) - 1(z-0) = 0$

OR $\boxed{x + y - z = 1}$

NORMAL LINE:

$x = 1 + t$, $y = t$, $z = -t$ OR $x - 1 = y = -z$

Turn over.

3. (2 points) Suppose that z is implicitly defined as a function of x and y by the following equation. Find $\partial z/\partial y$ and $\partial z/\partial x$.

$$\underbrace{x^3z + y^3z^3 - 9xyz + 4}_{F(x,y,z)} = 0$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(3y^2z^3 - 9xz)}{x^3 + 3y^3z^2 - 9xy}}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2z - 9yz)}{x^3 + 3y^3z^2 - 9xy}}$$

4. (1 point) Over a certain region in space the electrical potential V is given by

$$V(x, y, z) = 5x^2 - 3xy + xyz.$$

At the point $(3, 4, 5)$, in which direction does the potential increase most rapidly?

DIRECTION OF $\vec{\nabla} V(3,4,5)$ ...

$$\vec{\nabla} V(x,y,z) = (10x - 3y + yz)\hat{i} + (-3x + xz)\hat{j} + (xy)\hat{k}$$

$$\boxed{\vec{\nabla} V(3,4,5) = 38\hat{i} + 6\hat{j} + 12\hat{k}}$$

5. (2 points) Find and classify the critical points of $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$.

$f_x(x,y) = 2x - 2y - 2 = 0$

$f_y(x,y) = -2x + 4y + 2 = 0$

$x - y = 1$

$-x + 2y = -1$

$y = 0, \; x = 1$

$(1,0)$ IS THE ONLY CRITICAL PT

$$D = \begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = 8 - 4 = 4 > 0$$

$D > 0$ AND $f_{xx} > 0$

$\Downarrow$

$\boxed{f(1,0) = 0 \text{ IS A RELATIVE MIN.}}$