Math 233 - Quiz 6

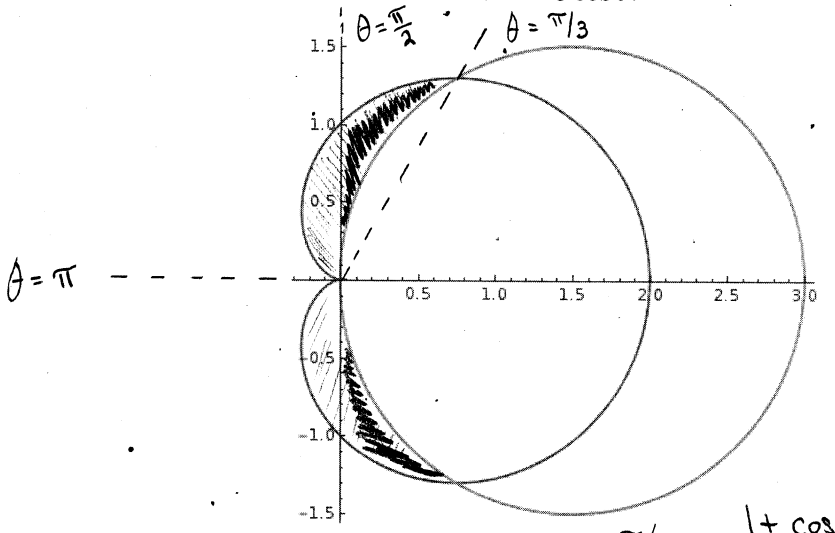
April 22, 2021

Name key

Score _____

Show all work to receive full credit. Supply explanations when necessary. This quiz is due April 29.

1. (5 points) Use a double integral to find the area of the polar region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.



$1 + \cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \frac{\pi}{3}$

DARK SHADED AREA $= 2 \cdot \int_{\pi/3}^{\pi/2} \int_{3\cos\theta}^{1+\cos\theta} r \, dr \, d\theta$

$= \int_{\pi/3}^{\pi/2} r^2 \Big|_{3\cos\theta}^{1+\cos\theta} d\theta = \int_{\pi/3}^{\pi/2} \left[(1+\cos\theta)^2 - 9\cos^2\theta \right] d\theta$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$= \int_{\pi/3}^{\pi/2} \left[1 + 2\cos\theta - 8\cos^2\theta \right] d\theta$

$= \int_{\pi/3}^{\pi/2} \left[1 + 2\cos\theta - 4(1+\cos 2\theta) \right] d\theta$

$= \int_{\pi/3}^{\pi/2} \left[-3 + 2\cos\theta - 4\cos 2\theta \right] d\theta = -3\theta + 2\sin\theta - 2\sin 2\theta \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

Turn over.

$= -\frac{3\pi}{2} + 2 - 0 + \pi - \sqrt{3} + \sqrt{3} = 2 - \frac{\pi}{2}$

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PROBLEM #1 CONTINUED...

$$\text{LIGHT SHADED AREA} = 2 \cdot \int_{\pi/2}^{\pi} \int_{0}^{1+\cos\theta} r \, dr \, d\theta$$

$$= \int_{\pi/2}^{\pi} (1+\cos\theta)^2 \, d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_{\pi/2}^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \int_{\pi/2}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \Big|_{\pi/2}^{\pi}$$

$$= \frac{3}{2}\pi + 0 + 0 - \frac{3\pi}{4} - 2 - 0$$

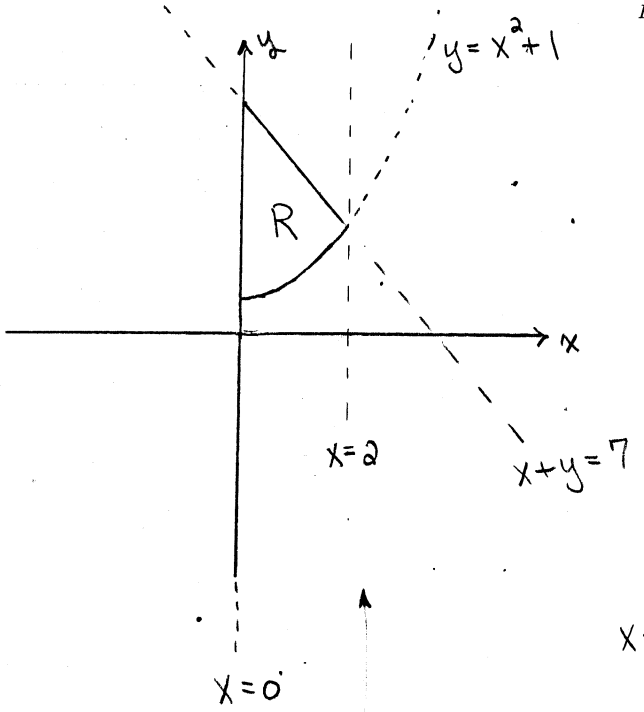
$$= \frac{3\pi}{4} - 2$$

$$\text{TOTAL AREA} = \left(2 - \frac{\pi}{2}\right) + \left(\frac{3\pi}{4} - 2\right) = \boxed{\frac{\pi}{4}}$$

SHADED DARK

SHADED LIGHT

2. (5 points) Let P be the space region in the first octant below the plane $x + y + z = 9$ and above the region in the xy-plane bounded by the graphs of $x = 0$, $y = x^2 + 1$ and $x + y = 7$. Evaluate the triple integral $\iiint_P dV$.



THE REGION P IS BOUNDED ABOVE BY $x+y+z=9$ AND BELOW BY R (IN PLANE $z=0$)

$$\iiint_P dV = \int_{x=0}^{x=2} \int_{y=x^2+1}^{y=7-x} \int_{z=0}^{z=9-x-y} dz\, dy\, dx$$

$y = x^2 + 1$

$x + y = 7$

$\Downarrow$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = 2$

$$= \int_0^2 \int_{x^2+1}^{7-x} (9-x-y)\, dy\, dx$$

$$= \int_0^2 9y - xy - \tfrac{1}{2}y^2 \Big|_{x^2+1}^{7-x} dx$$

$$= \int_0^2 \left[9(7-x) - x(7-x) - \tfrac{1}{2}(7-x)^2 - 9(x^2+1) + x(x^2+1) + \tfrac{1}{2}(x^2+1)^2\right] dx$$

$$= \int_0^2 \left(\tfrac{1}{2}x^4 + x^3 - \tfrac{15}{2}x^2 - 8x + 30\right) dx = \tfrac{1}{10}x^5 + \tfrac{1}{4}x^4 - \tfrac{15}{6}x^3 - 4x^2 + 30x \Big|_0^2$$

$$= \boxed{\frac{156}{5}}$$