

Math 233 - Test 2

March 11, 2021

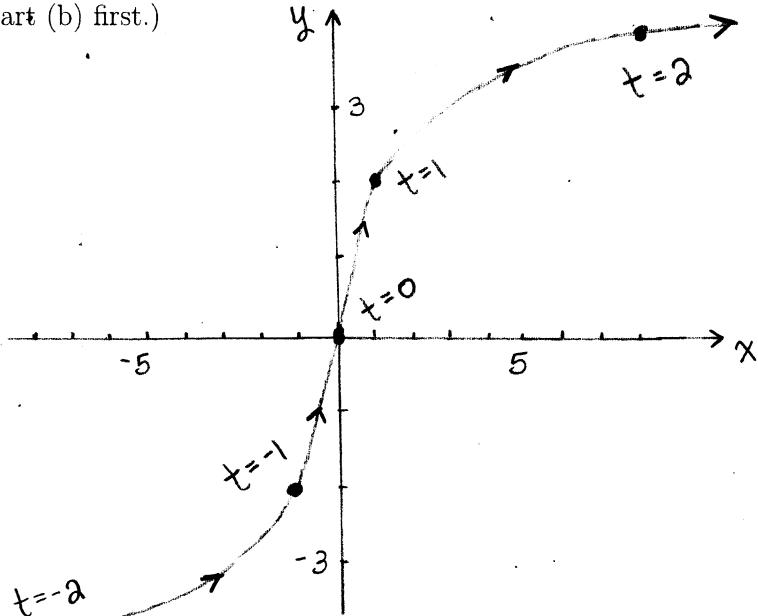
Name key _____
Score _____

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Consider the vector-valued function defined by $\vec{r}(t) = t^3\hat{i} + 2t\hat{j}$.

- (a) Sketch the graph of $\vec{r}(t)$. Use arrows to show the orientation of the curve.
(You may wish to do part (b) first.)

t	(x, y)
-2	(-8, -4)
-1	(-1, -2)
0	(0, 0)
1	(1, 2)
2	(8, 4)



- (b) Eliminate the parameter t to find an equation for the graph of \vec{r} in the rectangular coordinates x and y .

$$x = t^3$$

$$y = 2t \Rightarrow t = \frac{y}{2} \Rightarrow x = \frac{y^3}{8}$$

or
 $y^3 = 8x$

2. (3 points) What is the domain of $\vec{r}(t) = \frac{1}{2t+1}\hat{i} + \sqrt{t-3}\hat{j} + (\ln t)\hat{k}$?

$$\left. \begin{array}{l} x(t) : t \neq -\frac{1}{2} \\ y(t) : t \geq 3 \\ z(t) : t > 0 \end{array} \right\} \quad \begin{array}{l} \text{All together, domain of } \vec{r} \\ = \boxed{\{t : t \geq 3\}} \end{array}$$

3. (5 points) Find a vector-valued function whose graph is the line in space passing through the points $P(1, 2, -1)$ and $Q(2, 7, 5)$.

PARAMETRIC EQUATION'S :

$$\vec{r} = \vec{PQ} = \hat{i} + 5\hat{j} + 6\hat{k}$$

POINT P (1, 2, -1)

$$x = 1 + t$$

$$y = 2 + 5t$$

$$z = -1 + 6t$$

$$\vec{r}(t) = (1+t)\hat{i} + (2+5t)\hat{j}$$

$$+ (-1+6t)\hat{k}$$

4. (5 points) Let $\vec{r}(t) = \sqrt[3]{t}\hat{i} + \frac{1}{t+1}\hat{j} + e^{-t}\hat{k}$. Evaluate the definite integral $\int_0^1 \vec{r}(t) dt$.

$$\int_0^1 \left(t^{\frac{1}{3}}\hat{i} + \frac{1}{t+1}\hat{j} + e^{-t}\hat{k} \right) dt$$

$$= \left. \frac{3}{4}t^{\frac{4}{3}}\hat{i} + \ln(t+1)\hat{j} - e^{-t}\hat{k} \right|_0^1$$

$$= \frac{3}{4}\hat{i} + \ln(2)\hat{j} - (e^{-1} - 1)\hat{k}$$

$$= \left. \frac{3}{4}\hat{i} + \ln 2\hat{j} + \left(1 - \frac{1}{e}\right)\hat{k} \right|_0^1$$

5. (4 points) Let $\vec{r}(t) = 5\cos(\pi t)\hat{i} - 5\sin(\pi t)\hat{j}$. Show that $\vec{v}(t)$ is always orthogonal to $\vec{r}(t)$.

$$\vec{v}(t) = -5\pi \sin \pi t \hat{i} - 5\pi \cos \pi t \hat{j}$$

$$\vec{v}(t) \cdot \vec{r}(t) = -25\pi \cos \pi t \sin \pi t + 25\pi \sin \pi t \cos \pi t$$

$$= 0$$

DOT PROD = 0

$$\vec{v}(0) = \vec{0}$$

$$\vec{r}(0) = \hat{i} + 2\hat{j}$$

6. (8 points) At $t = 0$, an object starts from rest at the point $P(1, 2, 0)$ and moves with acceleration $\vec{a}(t) = \hat{j} + 2\hat{k}$. Find the location of the object at $t = 2$.

$$\vec{v}(t) = c_1 \hat{i} + (t + c_2) \hat{j} + (2t + c_3) \hat{k}$$

$$\vec{r}(t) = \hat{i} + \left(\frac{1}{2}t^2 + 2\right) \hat{j}$$

$$\vec{v}(0) = \vec{0} \Rightarrow c_1 = c_2 = c_3 = 0$$

$$+ t^2 \hat{k}$$

$$\vec{v}(t) = t \hat{j} + 2t \hat{k}$$

$$\vec{r}(t) = c_1 \hat{i} + \left(\frac{1}{2}t^2 + c_2\right) \hat{j} + (t^2 + c_3) \hat{k}$$

$$\vec{r}(0) = \hat{i} + 2\hat{j} \Rightarrow c_1 = 1, c_2 = 2, c_3 = 0$$

$$\vec{r}(2) = \hat{i} + 4\hat{j} + 4\hat{k}$$

(1, 4, 4)

7. (10 points) Let $\vec{r}(t) = 2 \sin(t) \hat{i} + 5t \hat{j} + 2 \cos(t) \hat{k}$.

- (a) Describe the graph of $\vec{r}(t)$.

THE GRAPH IS A HELIX WITH RADIUS 2,

CENTERED ON THE y-AXIS.

- (b) Find the principal unit normal vector for $\vec{r}(t)$.

$$\vec{r}'(t) = 2 \cos t \hat{i} + 5 \hat{j} - 2 \sin t \hat{k}$$

$$\|\vec{r}'(t)\| = \sqrt{4 \cos^2 t + 25 + 4 \sin^2 t} = \sqrt{29}$$

$$\hat{T}(t) = \frac{1}{\sqrt{29}} (2 \cos t \hat{i} + 5 \hat{j} - 2 \sin t \hat{k})$$

$$\hat{T}'(t) = \frac{1}{\sqrt{29}} (-2 \sin t \hat{i} - 2 \cos t \hat{k})$$

$$\|\hat{T}'(t)\| = \frac{2}{\sqrt{29}}$$

$$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = -\sin t \hat{i} - \cos t \hat{k}$$

8. (6 points) Parameterize the line described by $\vec{r}(t) = (3 - 3t)\hat{i} + 4t\hat{j}$ using the arc-length parameter s , from $t = 0$.

$$\vec{v}(t) = -3\hat{i} + 4\hat{j}$$

$$\|\vec{v}(t)\| = \sqrt{(-3)^2 + 4^2} = 5$$

$$s = \int_0^t 5 \, dr = 5t$$

$$t = \frac{s}{5}$$

$$\vec{r}(s) = \left(3 - \frac{3}{5}s\right)\hat{i} + \frac{4}{5}s\hat{j}$$

9. (8 points) A golf ball is hit in a horizontal direction with speed v_0 off the top edge of a building that is 100 ft tall.

- (a) Using $g = 32 \text{ ft/s}^2$ and ignoring air resistance, write the vector-valued function that gives the position of the golf ball at time t .

$$\begin{aligned}\vec{r}(t) &= v_0 \cos 0^\circ t \hat{i} + (-16t^2 + v_0 \sin 0^\circ t + 100)\hat{j} \\ &= v_0 t \hat{i} + (-16t^2 + 100)\hat{j}\end{aligned}$$

- (b) Find v_0 if the ball lands 450 ft away from the building.

$$v_0 t = 450$$

$$-16t^2 + 100 = 0$$

$$t^2 = \frac{100}{16}$$

$$t = \frac{10}{4} = 2.5$$

$$v_0 = \frac{450}{2.5} = 180 \text{ FT/SEC}$$

10. (3 points) Let $w(x, y, z) = x^2 + y^2 + z^2$. Describe, in detail, the level surface $w(x, y, z) = 9$.

$$x^2 + y^2 + z^2 = 9$$

Sphere centered at $(0, 0, 0)$

with radius 3.

11. (10 points) Consider the function defined by the equation $z = x^2 + y^2$.

(a) What is the domain of this function?

$x \in y$ CAN BE ANY REAL #'S \Rightarrow DOMAIN IS \mathbb{R}^2 .

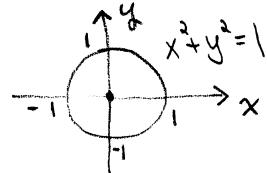
(b) What is the range of this function?

$$x^2 \geq 0 \text{ FOR ALL } x \quad \Rightarrow \quad \text{Range} = \{z : z \geq 0\}$$

$$y^2 \geq 0 \text{ FOR ALL } y$$

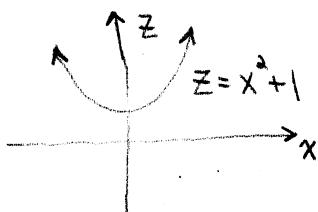
(c) Sketch the level curve $z = 1$.

$x^2 + y^2 = 1$ IS THE UNIT CIRCLE



(d) Sketch the level curve $y = 1$.

$z = x^2 + 1$ IS A PARABOLA



(e) Describe (or sketch) the graph of $z = x^2 + y^2$.

THE GRAPH IS A PARABOLOID WITH VERTEX AT $(0,0,0)$ OPENING UP THE Z-AXIS.

12. (8 points) Find k so that f is continuous everywhere.

$$f(x,y) = \begin{cases} \frac{x^4 - 4y^2}{x^2 + 2y^2}, & (x,y) \neq (0,0) \\ k, & (x,y) = (0,0) \end{cases}$$

$$k = \lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$$

$$\text{Along } y=0 : \lim_{x \rightarrow 0} \frac{x^4}{x^2} = \lim_{x \rightarrow 0} x^2 = 0$$

$$\text{Along } x=0 : \lim_{y \rightarrow 0} \frac{-4y^2}{2y^2} = -2$$

No limit \Rightarrow No such k .

13. (8 points) Consider the following limit: $\lim_{(x,y) \rightarrow (0,0)} \frac{xy + y^3}{x^2 + y^2}$.

(a) Show that the limit does not exist.

Along $x=0$:

$$\lim_{y \rightarrow 0} \frac{y^3}{y^2} = \lim_{y \rightarrow 0} y = 0$$

Along $y=x$:

$$\lim_{x \rightarrow 0} \frac{x^2 + x^3}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{x^2(1+x)}{2x^2} = \lim_{x \rightarrow 0} \frac{1+x}{2} = \frac{1}{2}$$

Two DIFFERENT LIMITS ALONG TWO PATHS \Rightarrow LIMIT DNE.

(b) Now think about this new limit: $\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y + y^3}{(x-1)^2 + y^2}$.

This limit also does not exist. What paths could you use to prove it?

THE PATHS CORRESPONDING TO THOSE ABOVE ARE

$x=1$ AND $y=x-1$, RESPECTIVELY.

14. (5 points) Let $z = \ln(xy + y^2)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = \frac{1}{xy + y^2} \cdot (y) = \frac{y}{xy + y^2} = \frac{1}{x+y}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{1}{x+y}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{xy + y^2} \cdot (x + 2y) \Rightarrow$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{x + 2y}{xy + y^2}}$$

15. (5 points) Let $f(x, y, z) = xyz$.

- (a) Before computing them, what do you expect to be true of f_{xyy} , f_{yxy} , and f_{yyx} ? Why?

THEY WILL ALL BE EQUAL. SINCE f IS A POLYNOMIAL,
ITS DERIVATIVES WILL BE POLYNOMIALS, AND THEREFORE,
CONTINUOUS. THE PARTIALS ARE MIXED THE SAME WAY
($2-y$'s, $1-x$) AND THEY ARE CONT. THEY MUST BE
EQUAL.

- (b) Compute each of the mixed partials mentioned in part (a).

$$f_x(x, y, z) = yz$$

$$f_{xy}(x, y, z) = z$$

$$f_{xxy}(x, y, z) = 0 = f_{yxy}(x, y, z) = f_{yyx}(x, y, z)$$

16. (4 points) Suppose you are given the function $f(x, y)$. Further suppose that $f_x(2, 1)$ and $f_y(2, 1)$ both exist. Explain what the number $f_x(2, 1)$ actually measures. How is that different from what the number $f_y(2, 1)$ measures?

$f_x(2, 1)$ = SLOPE OF THE TANGENT LINE AT $(2, 1)$
THAT IS LOOKING IN THE DIRECTION
PARALLEL TO THE X-AXIS.

$f_y(2, 1)$ IS SIMILARLY A SLOPE, BUT LOOKING
PARALLEL TO THE Y-AXIS.