Math 233 - Test 3a
April 15, 2021

Name <u>key</u>

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (5 points) Let $f(x,y) = x^2 + 5xy - y^2$. Use differentials to approximate the change in $f(x,y)$ as x changes from 2.00 to 1.97 and y changes from 3.00 to 3.05.

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$
$$= (2x+5y)\,dx + (5x-2y)\,dy$$

$$\Delta z \approx (2x+5y)\,\Delta x + (5x-2y)\,\Delta y$$

$$x = 2,\quad y = 3,\quad \Delta x = -0.03,\quad \Delta y = 0.05$$

$$\Rightarrow \quad \Delta z \approx (4+15)(-0.03) + (10-6)(0.05)$$

$$\boxed{\Delta z \approx -0.37}$$

2. (5 points) Assume that y is implicitly defined as a function of x by the equation $xe^y + ye^x = 2x^2y$. Use partial derivatives to find dy/dx.

Let $F(x,y) = xe^y + ye^x - 2x^2y$.

Then $\dfrac{dy}{dx} = \dfrac{-F_x}{F_y}$

$$\Downarrow$$

$$\boxed{\frac{dy}{dx} = \frac{-(e^y + ye^x - 4xy)}{xe^y + e^x - 2x^2}}$$

$\|\vec{v}\| = \sqrt{9+16} = 5$

3. (8 points) Find the directional derivative of $f(x,y) = \ln(x^2+y^2)$ at the point $P\,(1,2)$ in the direction of $\vec{v} = 3\hat{\imath} - 4\hat{\jmath}$.

$$\vec{\nabla} f(x,y) = \frac{2x}{x^2+y^2}\,\hat{\imath} + \frac{2y}{x^2+y^2}\,\hat{\jmath}$$

$$\vec{\nabla} f(1,2) = \frac{2}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$$

$$D_{\vec{v}} f(1,2) = \vec{\nabla} f(1,2) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{2}{5}\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right)\left(-\frac{4}{5}\right)$$

$$= -\frac{10}{25} = \boxed{-\frac{2}{5}}$$

Follow-up: If you were standing at the point P on the the graph of f looking in the direction of $\vec{v}$, would you be looking downhill or uphill? How do you know?

DOWNHILL. THE DIRECTIONAL DERIVATIVE IS NEGATIVE --- SURFACE SLOPES DOWNWARD.

4. (12 points) Find and classify the critical points of $g(x,y) = x^2 + x - 3xy + y^3 - 5$.

$$g_x(x,y) = 2x + 1 - 3y = 0$$
$$g_y(x,y) = -3x + 3y^2 = 0$$
$$\Downarrow$$
$$x = y^2$$
$$2y^2 - 3y + 1 = 0$$
$$(2y-1)(y-1) = 0$$
$$y = \tfrac{1}{2},\ y = 1$$

TWO CRIT. PTS:

$$\left(\tfrac{1}{4}, \tfrac{1}{2}\right), (1,1)$$

$$D(x,y) = \det\left(\begin{bmatrix} 2 & -3 \\ -3 & 6y \end{bmatrix}\right) = 12y - 9$$

$D\left(\frac{1}{4}, \frac{1}{2}\right) = -3 \Rightarrow$ $\left(\frac{1}{4}, \frac{1}{2}, -\frac{79}{16}\right)$ IS A SADDLE PT.

$D(1,1) = 3$, $f_{xx}(1,1) = 3$

$\Rightarrow$ $g(1,1) = -5$ IS A REL MIN.

5. (5 points) The electric voltage in a certain region in space is described by the function $V(x, y, z) = 5x^2 - 3xy + xyz$. At the point $(3, 4, 5)$, in what direction is the voltage increasing most rapidly? Give your answer as a unit vector.

DIRECTION OF gradient VECTOR.

$$\vec{\nabla} V(x,y,z) = (10x - 3y + yz)\,\hat{i} + (-3x + xz)\,\hat{j} + (xy)\,\hat{k}$$

$$\vec{\nabla} V(3,4,5) = 38\hat{i} + 6\hat{j} + 12\hat{k}$$

$$\|\vec{\nabla} V(3,4,5)\| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$$

$$\frac{19\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{406}}$$

6. (5 points) Suppose w is a function of x, y, z and x, y, z are functions of s, t. Write the chain rule formulas for $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$