Math 233 - Test 3b
April 15, 2021

Name key
Score _____

Show all work to receive full credit. Supply explanations where necessary. This test is due April 22.

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1. (6 points) Find a set of parametric equations for the line normal to the graph of $x^2 - 8xyz + y^2 + 6z^2 = 0$ at the point $P(1,1,1)$.

$F(x,y,z) = x^2 - 8xyz + y^2 + 6z^2$

$\vec{\nabla} F$ IS NORMAL TO $F(x,y,z) = 0$ AT $(1,1,1)$.

$\vec{\nabla} F(x,y,z) = (2x - 8yz)\hat{i} + (2y - 8xz)\hat{j} + (12z - 8xy)\hat{k}$

$\vec{n} = \vec{\nabla} F(1,1,1) = -6\hat{i} - 6\hat{j} + 4\hat{k}$

NORMAL LINE:

$x = -6t + 1$
$y = -6t + 1$
$z = 4t + 1$

2. (6 points) Let $f(x,y) = \sqrt{20 - x^2 - 7y^2}$. Find the linearization of f at $(2,1)$ and use it to approximate $f(1.98, 1.04)$. (The textbook uses the name *linear approximation* instead of linearization.)

$L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f_x(x,y) = \frac{1}{2}(20 - x^2 - 7y^2)^{-1/2}(-2x) \Rightarrow f_x(2,1) = \frac{-2}{3}$

$f_y(x,y) = \frac{1}{2}(20 - x^2 - 7y^2)^{-1/2}(-14y) \Rightarrow f_y(2,1) = -\frac{7}{3}$

$f(2,1) = \sqrt{9} = 3$

$L(x,y) = 3 - \frac{2}{3}(x-2) - \frac{7}{3}(y-1)$

$f(1.98, 1.04) \approx 3 - \frac{2}{3}(-0.02) - \frac{7}{3}(0.04)$

$= \frac{73}{25} = 2.92$

$g_x(x,y) = 3x^2 + 2y$, $g_y(x,y) = 2x - 1$

3. (8 points) Use the definition of differentiability to show that $g(x,y) = x^3 + 2xy - y$ is differentiable everywhere in $\mathbb{R}^2$.

$$g(x+\Delta x, y+\Delta y) - g(x,y) = \left[(x+\Delta x)^3 + 2(x+\Delta x)(y+\Delta y) - (y+\Delta y)\right] - \left[x^3 + 2xy - y\right]$$

$$= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2xy + 2x\Delta y + 2y\Delta x + 2\Delta x\Delta y - y - \Delta y$$
$$- x^3 \qquad\qquad - 2xy \qquad\qquad + y$$

$$= 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2x\Delta y + 2y\Delta x + 2\Delta x\Delta y - \Delta y$$

$$= (3x^2 + 2y)\Delta x + (2x-1)\Delta y + (3x\Delta x + \Delta x^2)\Delta x + (2\Delta x)\Delta y$$

$$= g_x(x,y)\,\Delta x + g_y(x,y)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$

AND $\epsilon_1 = 3x\Delta x + \Delta x^2 \longrightarrow 0$

$\epsilon_2 = 2\Delta x \longrightarrow 0$

AS $(\Delta x, \Delta y) \to (0,0)$.

BY DEFINITION, g IS DIFFERENTIABLE EVERYWHERE.

4. (5 points) Find a vector that is normal to the surface $z = x^2 - 2xy + 5y^2 + 3x - 2y + 4$ at the point $(1,1,9)$.

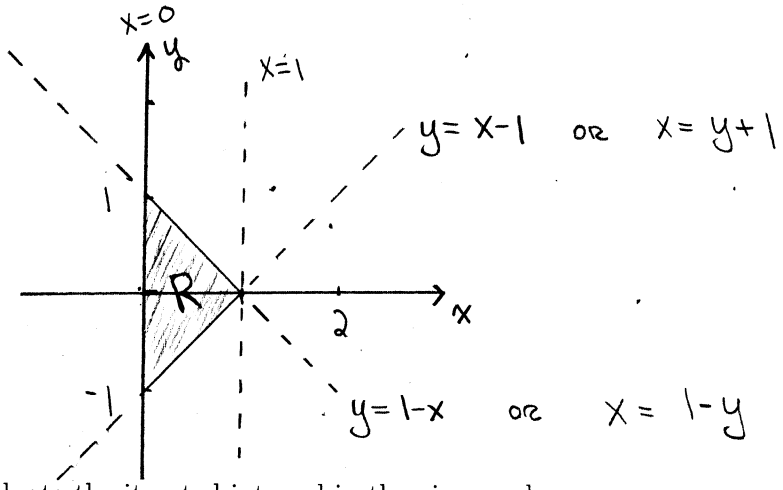
$$F(x,y,z) = x^2 - 2xy + 5y^2 + 3x - 2y + 4 - z$$

$$\vec{\nabla} F(x,y,z) = (2x - 2y + 3)\hat{i} + (-2x + 10y - 2)\hat{j} - \hat{k}$$

$$\boxed{\vec{\nabla} F(1,1,9) = 3\hat{i} + 6\hat{j} - \hat{k}}$$

5. (15 points) Consider the iterated integral $\int_0^1 \int_{x-1}^{1-x} x \, dy \, dx$.

(a) Sketch the region of integration.



(b) Evaluate the iterated integral in the given order.

$$\int_0^1 xy \Big|_{y=x-1}^{y=1-x} dx = \int_0^1 x(1-x) - x(x-1) \, dx$$

$$= \int_0^1 (2x - 2x^2) \, dx = x^2 - \frac{2}{3}x^3 \Big|_0^1 = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

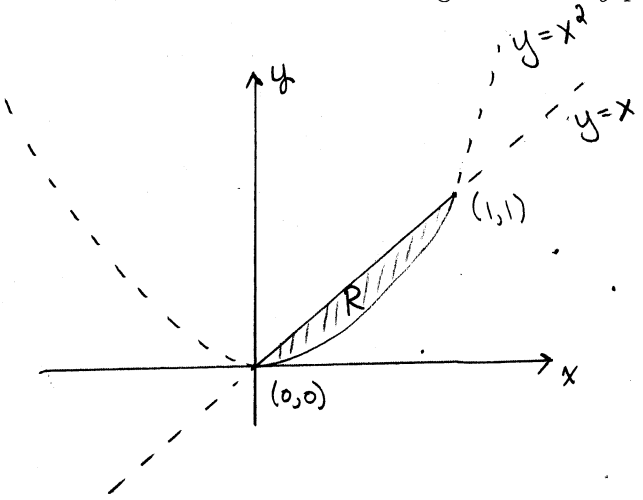
(c) Reverse the order of integration and write the new equivalent iterated integral(s). DO NOT EVALUATE.

Need two integrals!

$$\int_{-1}^{0} \int_{0}^{y+1} x \, dx \, dy$$

$$+ \int_0^1 \int_0^{1-y} x \, dx \, dy$$

6. (10 points) Find the volume of the solid region under the surface $z = 2x + y^2$ and above the bounded region in the xy-plane between the graphs of $y = x^2$ and $y = x$.



$$\iint_R (2x+y^2)\,dA$$

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (2x+y^2)\,dy\,dx$$

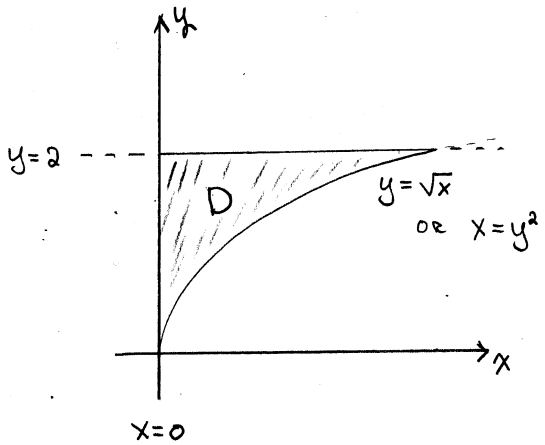
$$= \int_0^1 2xy + \tfrac{1}{3}y^3 \Big|_{x^2}^{x} dx = \int_0^1 \left(2x^2 + \tfrac{1}{3}x^3 - 2x^3 - \tfrac{1}{3}x^6\right) dx = \tfrac{2}{3} + \tfrac{1}{12} - \tfrac{2}{4} - \tfrac{1}{21}$$

$$= \boxed{\tfrac{17}{84}}$$

7. (10 points) Consider the double integral

$$\iint_D \frac{3}{1+y^2}\,dA,$$

where D is the region in the xy-plane bounded by the graphs of $x = 0$, $y = \sqrt{x}$, and $y = 2$. Write as an iterated integral and evaluate. (Use whichever order of integration you prefer.)



$$\int_{y=0}^{y=2} \int_{x=0}^{x=y^2} \frac{3}{1+y^2}\,dx\,dy$$

$$= \int_0^2 \frac{3x}{1+y^2} \Big|_0^{y^2} dy = 3\int_0^2 \frac{y^2}{1+y^2}\,dy$$

$$= 3\int_0^2 \left(1 - \frac{1}{1+y^2}\right) dy = 3\left(y - \tan^{-1} y\right)\Big|_0^2$$

$$= \boxed{6 - 3\tan^{-1} 2 \approx 2.678554}$$

$$\frac{y^2}{1+y^2} = \frac{y^2+1-1}{1+y^2} = 1 - \frac{1}{1+y^2}$$