Math 233 - Quiz 4

February 17, 2022

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. This quiz is due February 22.

1. (2 points) A particle is moving along the graph of $\vec{r}(t) = te^{-t}\,\hat{\imath} + t^2\,\hat{\jmath} + 2t\,\hat{k}$. Set up the definite integral that gives the length of the path of the particle over the interval $1 \le t \le 2$. Use technology to approximate the value of your integral.

$$\vec{r}\,'(t) = \left(-te^{-t} + e^{-t}\right)\hat{\imath} + (2t)\hat{\jmath} + 2\hat{k}$$

$$\int_1^2 \|\vec{r}\,'(t)\|\,dt = \int_1^2 \sqrt{\left(-te^{-t}+e^{-t}\right)^2 + 4t^2 + 4}\;dt$$

$$\approx 3.6216$$

2. (2 points) Let $\vec{r}(t) = -6t^2\,\hat{\imath} + (\frac{5}{2}t^2 + 3)\,\hat{\jmath} + 7\,\hat{k}$ for $t \ge 1$. Reparameterize $\vec{r}$ in terms of the arc-length parameter.

$$\vec{r}\,'(t) = -12t\,\hat{\imath} + 5t\,\hat{\jmath} + 0\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{144t^2 + 25t^2} = \sqrt{169t^2} = 13|t| = 13t \text{ For } t \ge 1$$

$$s = \int_1^t 13u\,du = \frac{13}{2}u^2\Big|_1^t = \frac{13}{2}t^2 - \frac{13}{2} \Rightarrow t^2 = \frac{2}{13}s + 1$$

$$t = \sqrt{\frac{2}{13}s + 1}\,, \quad s \ge 0$$

Turn over.

$$\vec{r}(s) = -6\left(\frac{2}{13}s+1\right)\hat{\imath} + \left[\frac{5}{2}\left(\frac{2}{13}s+1\right) + 3\right]\hat{\jmath} + 7\hat{k}, \quad s \ge 0$$

3. (2 points) Determine the curvature of the graph of $y = \ln x$ at the point where $x = 1$.

$y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$

$$K = \frac{\frac{1}{x^2}}{\left[1 + \frac{1}{x^2}\right]^{3/2}}$$

$$K(x=1) = \frac{1}{2^{3/2}} \approx 0.3536$$

4. (2 points) Determine the curvature of the graph of $\vec{r}(t) = (2\cos t)\,\hat{\imath} + (3\sin t)\,\hat{\jmath}$ at the point $(0, 3)$ (that is, the point where $t = \pi/2$).

$\vec{r}\,'(t) = -2\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath}$

$\vec{r}\,''(t) = -2\cos t\,\hat{\imath} - 3\sin t\,\hat{\jmath}$

$\vec{r}\,'(\frac{\pi}{2}) = -2\hat{\imath}$

$\vec{r}\,''(\frac{\pi}{2}) = -3\hat{\jmath}$

$$\vec{r}\,'(\tfrac{\pi}{2}) \times \vec{r}\,''(\tfrac{\pi}{2}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = 6\hat{k}$$

$$K(t = \tfrac{\pi}{2}) = \frac{\|\vec{r}\,'(\frac{\pi}{2}) \times \vec{r}\,''(\frac{\pi}{2})\|}{\|\vec{r}\,'(\frac{\pi}{2})\|^3} = \frac{6}{8} = \boxed{\frac{3}{4}}$$

5. (2 points) Let $\vec{r}(t) = 2t\,\hat{\imath} + 4t^2\,\hat{\jmath}$. Compute $\hat{N}(1)$.

$\vec{r}\,'(t) = 2\hat{\imath} + 8t\hat{\jmath}$

$$\hat{T}(t) = \frac{2\hat{\imath} + 8t\hat{\jmath}}{\sqrt{4 + 64t^2}} = \frac{\hat{\imath} + 4t\hat{\jmath}}{\sqrt{1 + 16t^2}}$$

$$\hat{T}\,'(t) = \frac{\sqrt{1+16t^2}\,(4\hat{\jmath}) - (\hat{\imath} + 4t\hat{\jmath})(\frac{1}{2})(1+16t^2)^{-1/2}(32t)}{1 + 16t^2}$$

$$\hat{T}\,'(1) = \frac{\sqrt{17}\,(4\hat{\jmath}) - (\hat{\imath} + 4\hat{\jmath})(17)^{-1/2}(16)}{17}$$

$$= \frac{68\hat{\jmath} - 16\hat{\imath} - 64\hat{\jmath}}{17\sqrt{17}} = \frac{-16\hat{\imath} + 4\hat{\jmath}}{17\sqrt{17}}$$

$$\hat{N}(1) = \frac{\hat{T}\,'(1)}{\|\hat{T}\,'(1)\|} = \boxed{\frac{-4\hat{\imath} + \hat{\jmath}}{\sqrt{17}}}$$