Math 233 - Quiz 7

March 24, 2022

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. This quiz is due March 29.

1. (2 points) Use differentials to estimate the change in $f(x, y, z) = x^2 \ln(5yz + 1)$ as (x, y, z) changes from $(2, 1, 3)$ to $(1.99, 1.02, 3.05)$.

$$\Delta z \approx f_x \,\Delta x + f_y \,\Delta y + f_z \,\Delta z$$

$$\approx 2x \ln(5yz+1) \cdot \Delta x + \frac{5x^2 z}{5yz+1} \Delta y + \frac{5x^2 y}{5yz+1} \Delta z$$

$$(x,y,z) = (2,1,3), \quad (\Delta x, \Delta y, \Delta z) = (-0.01, 0.02, 0.05)$$

$$\Delta z \approx (4 \ln 16)(-0.01) + \frac{60}{16}(0.02) + \frac{20}{16}(0.05)$$

$$= -0.04 \ln 16 + \frac{2.2}{16} \approx 0.0266$$

2. (3 points) Use the **definition of differentiability** to show that $f(x, y) = y^2 - xy + 5x$ is differentiable everywhere on $\mathbb{R}^2$.

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = \left[(y+\Delta y)^2 - (x+\Delta x)(y+\Delta y) + 5(x+\Delta x)\right] - \left[y^2 - xy + 5x\right]$$

$$= \cancel{y^2} + \underline{2y\Delta y} + \underline{\Delta y^2} - \cancel{xy} - y\Delta x - \underline{x\Delta y} - \underline{\Delta x \Delta y} + \cancel{5x} + \underline{5\Delta x} - \cancel{y^2} + \cancel{xy} - \cancel{5x}$$

$$= (-y+5)\Delta x + (2y-x)\Delta y + (-\Delta y)\Delta x + (\Delta y)\Delta y$$

$$= f_x(x,y)\,\Delta x + f_y(x,y)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

WHERE $\epsilon_1 = -\Delta y$ AND $\epsilon_2 = \Delta y$.

SINCE Δz HAS THE CORRECT FORM AND $\epsilon_1 \to 0$ AND $\epsilon_2 \to 0$ AS $(\Delta x, \Delta y) \to (0,0)$,

Turn over.

$\therefore$ IT FOLLOWS THAT f IS DIFFERENTIABLE.

3. (2.5 points) Find the linearization of $f(x, y) = \sqrt{1 + x - y^2}$ at $(4, 1)$, and then use it to approximate $f(3.96, 1.02)$.

$$f_x(x,y) = \frac{1}{2}(1+x-y^2)^{-1/2}, \quad f_x(4,1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f_y(x,y) = \frac{1}{2}(1+x-y^2)^{-1/2}(-2y), \quad f_y(4,1) = \frac{-1}{\sqrt{4}} = -\frac{1}{2}$$

$$f(4,1) = \sqrt{4} = 2$$

$$\boxed{L(x,y) = 2 + \frac{1}{4}(x-4) - \frac{1}{2}(y-1)}$$

$$f(3.96, 1.02) \approx L(3.96, 1.02) = 2 + \frac{1}{4}(-0.04) - \frac{1}{2}(0.02)$$

$$= 2 - 0.01 - 0.01 = \boxed{1.98}$$

4. (2.5 points) Find an equation of the plane tangent to the graph of $g(x, y) = 3e^{y^2 - x^2}$ at the point where $(x, y) = (2, 2)$.

$$g_x(x,y) = (3e^{y^2-x^2})(-2x), \quad g_x(2,2) = (3e^0)(-4) = -12$$

$$g_y(x,y) = (3e^{y^2-x^2})(2y), \quad g_y(2,2) = (3e^0)(4) = 12$$

$$g(2,2) = 3$$

$$z = 3 - 12(x-2) + 12(y-2)$$

OR

$$\boxed{12x - 12y + z = 3}$$