Math 233 - Test 2
March 10, 2022

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (9 points) A curve is defined by the vector-valued function shown below. Starting from $t = 0$, reparameterize the curve in terms of the arc-length parameter.

$$\vec{r}(t) = (7t - 6)\,\hat{\imath} + (4t + 1)\,\hat{\jmath} + (4t + 3)\,\hat{k}.$$

Follow-up: Once you have reparameterized, compute $\|\vec{r}\,'(s)\|$.

$\vec{r}\,'(t) = 7\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$, $\|\vec{r}\,'(t)\| = \sqrt{49+16+16} = \sqrt{81} = 9$

$S(t) = \int_0^t 9\,du = 9u\Big|_0^t = 9t$

$S = 9t \Rightarrow t = \frac{s}{9} \Rightarrow \boxed{\vec{r}(s) = \left(\frac{7}{9}s - 6\right)\hat{\imath} + \left(\frac{4}{9}s + 1\right)\hat{\jmath} + \left(\frac{4}{9}s + 3\right)\hat{k}}$

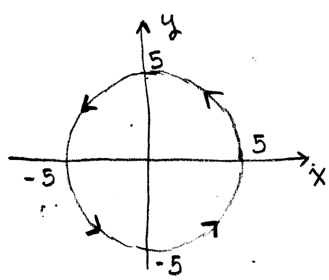
$\vec{r}\,'(s) = \frac{7}{9}\hat{\imath} + \frac{4}{9}\hat{\jmath} + \frac{4}{9}\hat{k} = \frac{1}{9}\vec{r}\,'(t)$

$\Rightarrow \boxed{\|\vec{r}\,'(s)\| = 1}$

2. (5 points) Sketch, or describe in detail, a 2-dimensional curve whose curvature is constant and nonzero. Then say what the curvature of your curve actually is and how you know.

A CIRCLE OF RADIUS a HAS CONSTANT CURVATURE $\kappa = \frac{1}{a}$.

SO SPECIFICALLY, THE CIRCLE DESCRIBED BY $\vec{r}(t) = 5\cos t\,\hat{\imath} + 5\sin t\,\hat{\jmath}$



HAS $\kappa = \frac{1}{5}$.

3. (16 points) A cannonball is fired toward the sea from a 192-ft cliff. The cannon is aimed at an angle of 30° above the horizontal and the initial speed of the cannonball is 128 ft/sec. (Use $g = 32\,\text{ft/sec}^2$ for this problem.)

(a) Determine the function that gives the position of the cannonball at time t. .

$$\vec{r}(t) = 128 \cos 30^\circ\, t\, \hat{\imath} + \left(-16t^2 + 128 \sin 30^\circ\, t + 192\right) \hat{\jmath}$$

$$\vec{r}(t) = 64\sqrt{3}\, t\, \hat{\imath} + \left(-16t^2 + 64t + 192\right) \hat{\jmath}$$

(b) When will the cannonball splash into the sea?

$$-16t^2 + 64t + 192 = 0$$

$$-16\left(t^2 - 4t - 12\right) = 0$$

$$-16(t-6)(t+2) = 0 \Rightarrow \boxed{t = 6 \text{ SECONDS}}$$

(c) How far out to sea will the cannonball hit the water?

RANGE =

$$(64\sqrt{3})(6) = 384\sqrt{3} \approx \boxed{665 \text{ FT}}$$

(d) Set up the definite integral that gives the total length of the cannonball's path. Use your calculator to estimate the value of the integral.

$$\vec{r}\,'(t) = 64\sqrt{3}\,\hat{\imath} + (-32t + 64)\,\hat{\jmath}$$

$$\text{LENGTH} = \int_0^6 \sqrt{(64\sqrt{3})^2 + (-32t + 64)^2}\, dt \approx 761.6 \text{ FT}$$

4. (8 points) Consider the function $f(x, y) = \ln(4 - x - y)$.

(a) Evaluate $f(2, 1)$.

$$f(2,1) = \ln(4-2-1) = \ln(1) = \boxed{0}$$

(b) What is the domain of f?

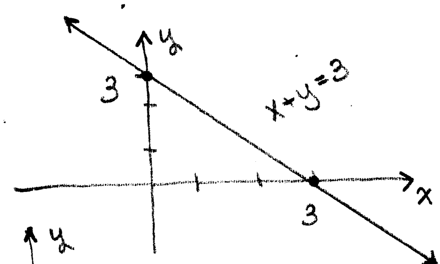
$$4-x-y > 0 \Rightarrow \boxed{\text{DOMAIN} = \{(x,y) : x+y < 4\}}$$

(c) What is the range of f?

$$\boxed{\mathbb{R}}$$

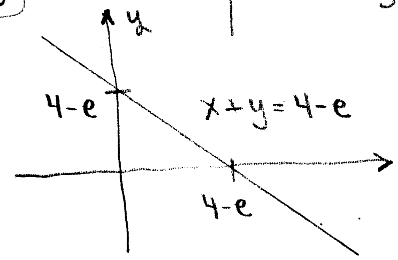
(d) Sketch the level curve $f(x, y) = 0$.

$$f(x,y) = 0 \Rightarrow 4-x-y=1$$
$$\boxed{x+y=3}$$



(e) Sketch the level curve $f(x, y) = 1$.

$$f(x,y) = 1 \Rightarrow 4-x-y = e$$
$$\boxed{x+y=4-e}$$



5. (8 points) Let $G(x, y, z) = \sqrt{x^2 + y^2 - z}$.

(a) Evaluate $G(-4, 5, -8)$.

$$G(-4,5,-8) = \sqrt{16+25+8} = \sqrt{49} = \boxed{7}$$

(b) What is the domain of G?

$$x^2+y^2-z \geq 0 \Rightarrow z \leq x^2+y^2 \qquad \boxed{\text{DOMAIN} = \{(x,y,z) : z \leq x^2+y^2\}}$$

(c) Describe or sketch the level surface $G(x, y, z) = 0$.

$$x^2+y^2-z = 0 \Rightarrow \boxed{z = x^2+y^2}$$ THE SURFACE IS A PARABOLOID OPENING UP THE Z-AXIS W/ VERTEX (0,0,0).

(d) Describe or sketch the level surface $G(x, y, z) = -1$.

NOT POSSIBLE. NO SUCH LEVEL SURFACE.

(e) Describe or sketch the level surface $G(x, y, z) = 1$.

$$x^2+y^2-z=1 \Rightarrow x^2+y^2-1 = z$$

SAME PARABOLOID AS IN PART (c), BUT SHIFTED DOWN 1 UNIT.

6. (20 points) Determine the limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(3,3)} \frac{x^2 - y^2 + x - y}{2x - 2y}$ % More work

$$\lim_{(x,y)\to(3,3)} \frac{(x-y)(x+y)+(x-y)}{2(x-y)} = \lim_{(x,y)\to(3,3)} \frac{x+y+1}{2} = \boxed{\frac{7}{2}}$$

(b) $\displaystyle\lim_{(x,y,z)\to(1,2,-1)} \frac{2x - 3y - 4z}{x + y^2 - z} = \frac{2(1)-3(2)-4(-1)}{(1)+(2)^2-(-1)} = \frac{2-6+4}{1+4+1} = \boxed{0}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ % More work.

Along $x=0$:

$$\lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

Along $y=x$:

$$\lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

$\boxed{\text{Limit DNE}}$

(d) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x - y}{\sqrt{x} - \sqrt{y}}$ % More work.

$$\lim_{(x,y)\to(1,1)} \frac{x-y}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(1,1)} \frac{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}$$

$$= \lim_{(x,y)\to(1,1)} \left(\sqrt{x}+\sqrt{y}\right) = \boxed{2}$$

7. (8 points) Let $f(x,y,z) = x^2y^3 + 2xyz - 3yz$.

(a) Compute $f_x(-2,1,2)$.

$f_x(x,y,z) = 2xy^3 + 2yz \Rightarrow f_x(-2,1,2) = 2(-2)(1)^3 + 2(1)(2)$

$= \boxed{0}$

(b) Which is (slightly) simpler to compute f_{xy} or f_{yx}? Why?

I THINK f_{xy} -- X FIRST, THEN y.

AFTER 1ST DERIVATIVE, f_x HAS ONLY TWO TERMS, WHEREAS f_y HAS THREE TERMS.

(c) Do you expect that $f_{xy}(x,y) = f_{yx}(x,y)$? Why?

YES, f IS A POLYNOMIAL, SO f_{xy} AND f_{yx} WILL BE CONTINUOUS. $\therefore$ BY OUR THEOREM, THEY'LL BE EQUAL.

(d) Compute $f_{yxz}(x,y,z)$.

↳ I'LL DO $f_{xzy}(x,y,z)$ INSTEAD.

$f_x = 2xy^3 + 2yz$, $f_{xz} = 2y$, $\boxed{f_{xzy} = 2}$

8. (5 points) Consider the function $f(x,y) = (x^2+y^2)^{1/3}$. Would you expect $f_{xy}(0,0)$ and $f_{yx}(0,0)$ to be equal? Explain your reasoning.

NO, NOT NECESSARILY. f_{xy} AND f_{yx} WILL HAVE x^2+y^2 IN THEIR DENOMINATORS, SO I DON'T EXPECT THEM TO BE CONTINUOUS (OR EVEN DEFINED) AT $(0,0)$.

9. (5 points) If you stood on the graph of $z = \ln(xy^2 - y + 1)$ at the point $(x,y) = (1,-1)$ and looked in the direction of the positive y-axis, would you be looking uphill or downhill? Show your work.

$$\frac{\partial z}{\partial y} = \frac{2xy - 1}{xy^2 - y + 1}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(1,-1)} = \frac{-3}{3} = -1$$

DOWNHILL!

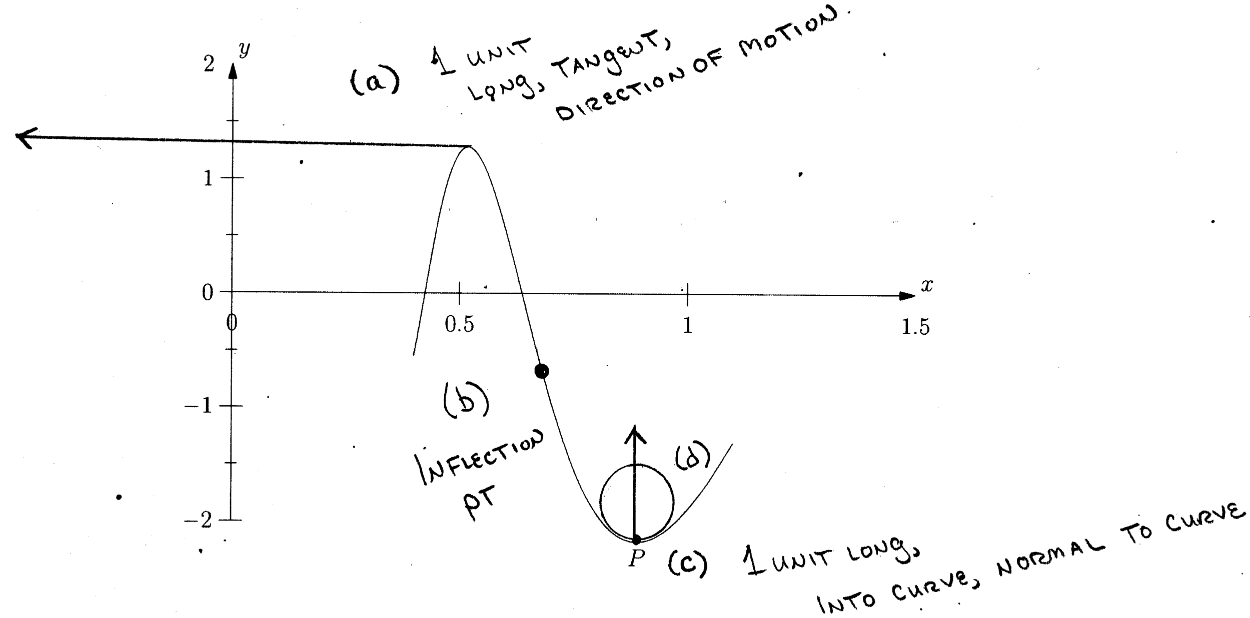
10. (6 points) Suppose a particle moves along the curve from **right to left**. Sketch and label each of the following. Make note of the scale.

(a) The unit tangent vector at the point of greatest curvature

(b) A point where the principal unit normal vector does not exist

(c) The principal unit normal vector at the point P

(d) (1 pt ex cred) The circle of curvature at the P



11. (10 points) Let $\vec{r}(t) = \sin 3t\,\hat{\imath} - \cos 3t\,\hat{\jmath} + 3t\,\hat{k}$. Compute $\hat{N}(t)$.

$$\vec{r}\,'(t) = 3\cos 3t\,\hat{\imath} + 3\sin 3t\,\hat{\jmath} + 3\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{\underbrace{9\cos^2 3t + 9\sin^2 3t}_{9} + 9} = \sqrt{18} = 3\sqrt{2}$$

$$\hat{T}(t) = \frac{1}{\sqrt{2}}\left(\cos 3t\,\hat{\imath} + \sin 3t\,\hat{\jmath} + \hat{k}\right)$$

$$\hat{T}'(t) = \frac{1}{\sqrt{2}}\left(-3\sin 3t\,\hat{\imath} + 3\cos 3t\,\hat{\jmath}\right)$$

$\hat{N}(t) =$ NORMALIZED $\hat{T}'(t) =$

$$\boxed{-\sin 3t\,\hat{\imath} + \cos 3t\,\hat{\jmath}}$$