Math 233 - Test 3a
April 14, 2022

Name key
Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) The volume V of a right circular cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. Suppose that the height decreases from 20 in to 19.95 in, while the radius increases from 4 in to 4.06 in. Use differentials to approximate the change in volume.

$$dV = \frac{2}{3}\pi r h \, dr + \frac{1}{3}\pi r^2 \, dh$$

$$\Delta V \approx \frac{2}{3}\pi r h \Delta r + \frac{1}{3}\pi r^2 \Delta h = \frac{1}{3}\pi r(2h\Delta r + r\Delta h)$$

$$h = 20, \ \Delta h = -0.05, \ r = 4, \ \Delta r = 0.06$$

$$\Delta V \approx 9.22 \text{ in}^3$$

2. (6 points) Find the directional derivative of $f(x,y) = \tan^{-1}(y/x)$ at the point $(-2,2)$ in the direction of $\vec{v} = -\hat{\imath} - \hat{\jmath}$.

$$\vec{\nabla} f(x,y) = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2}\hat{\imath} + \frac{1/x}{1+\left(\frac{y}{x}\right)^2}\hat{\jmath}$$

$$\vec{\nabla} f(-2,2) = \frac{-1/2}{2}\hat{\imath} + \frac{-1/2}{2}\hat{\jmath} = -\frac{1}{4}\hat{\imath} - \frac{1}{4}\hat{\jmath}$$

$$\|\vec{v}\| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$$

$$\frac{\vec{\nabla} f(-2,2)\cdot\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}}\left(\frac{1}{4}+\frac{1}{4}\right) = \boxed{\frac{1}{2\sqrt{2}}}$$

3. (6 points) Let $z = x^2y^2 - x + 2y$, where $x = \sqrt{u}$ and $y = uv^3$. Use the appropriate chain rule to compute $\partial z/\partial u$ when $(u, v) = (1, -2)$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2xy^2 - 1)\left(\tfrac{1}{2}u^{-1/2}\right) + (2x^2y + 2)(v^3)$$

$u=1,\ v=-2,\ x=1,\ y=-8$

$$\left.\frac{\partial z}{\partial u}\right|_{\substack{u=1\\v=-2}} = (127)\left(\tfrac{1}{2}\right) + (-14)(-8)$$

$$= \boxed{175.5}$$

OPPOSITE ∇G.

4. (6 points) Let $G(x, y, z) = \dfrac{x}{z} + \dfrac{z}{y^2}$. Find a unit vector in the direction in which G decreases most rapidly at $P(1, 2, -2)$. What is the corresponding rate of decrease?

$$\vec{\nabla} G(x,y,z) = \left(\frac{1}{z}\right)\hat{i} + \left(\frac{-2z}{y^3}\right)\hat{j} + \left(-\frac{x}{z^2} + \frac{1}{y^2}\right)\hat{k}$$

$$-\vec{\nabla} G(1,2,-2) = -\left(\frac{1}{-2}\right)\hat{i} - \left(\frac{4}{8}\right)\hat{j} - \left(-\frac{1}{4} + \frac{1}{4}\right)\hat{k}$$

$$= \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}$$

$\boxed{\text{UNIT VECTOR IS } \frac{1}{\sqrt{2}}(\hat{i} - \hat{j})}$

RATE OF DECREASE:

$$= -\|\vec{\nabla} G\| = \boxed{-\frac{\sqrt{2}}{2}}$$

5. (6 points) Find the linearization of $f(x, y) = x^{1/3}y^{1/2}$ at $(x, y) = (8, 9)$. Then use your linearization to approximate $f(7.87, 9.09)$.

$f_x(x,y) = \frac{1}{3}x^{-2/3}y^{1/2}$, $f_x(8,9) = \frac{1}{3}\left(\frac{1}{4}\right)(3) = \frac{1}{4}$

$f(8,9) = 6$

$f_y(x,y) = \frac{1}{2}x^{1/3}y^{-1/2}$, $f_y(8,9) = \frac{1}{2}(2)\left(\frac{1}{3}\right) = \frac{1}{3}$

$$\boxed{L(x,y) = 6 + \frac{1}{4}(x-8) + \frac{1}{3}(y-9)}$$

$$f(7.87, 9.09) \approx L(7.87, 9.09)$$

$$= 6 + \frac{1}{4}(-0.13) + \frac{1}{3}(0.09)$$

$$= \boxed{5.9975}$$

6. (8 points) Find an equation of the plane tangent to the surface $\sin(xz) = 4\cos(yz)$ at the point $(\pi, \pi/2, 1)$.

$$F(x,y,z) = \sin(xz) - 4\cos(yz)$$

OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$.

$$\vec{\nabla} F(x,y,z) = z\cos(xz)\,\hat{\imath} + 4z\sin(yz)\,\hat{\jmath} + \left(x\cos(xz) + 4y\sin(yz)\right)\hat{k}$$

$$\vec{n} = \vec{\nabla} F(\pi, \pi/2, 1) = -\hat{\imath} + 4\hat{\jmath} + (-\pi + 2\pi)\,\hat{k} = -\hat{\imath} + 4\hat{\jmath} + \pi\hat{k}$$

TANGENT PLANE: $-(x-\pi) + 4\left(y - \frac{\pi}{2}\right) + \pi(z-1) = 0$

OR

$-x + 4y + \pi z = 2\pi$

7. (6 points) Find and classify the critical point(s) of $f(x,y) = x^2 + xy + y^2 - 3x$.

$f_x(x,y) = 2x + y - 3$

$f_y(x,y) = x + 2y$

$$d = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$x + 2y = 0 \Rightarrow x = -2y$

$2x + y = 3 \qquad -3y = 3$

$y = -1$

$x = 2$

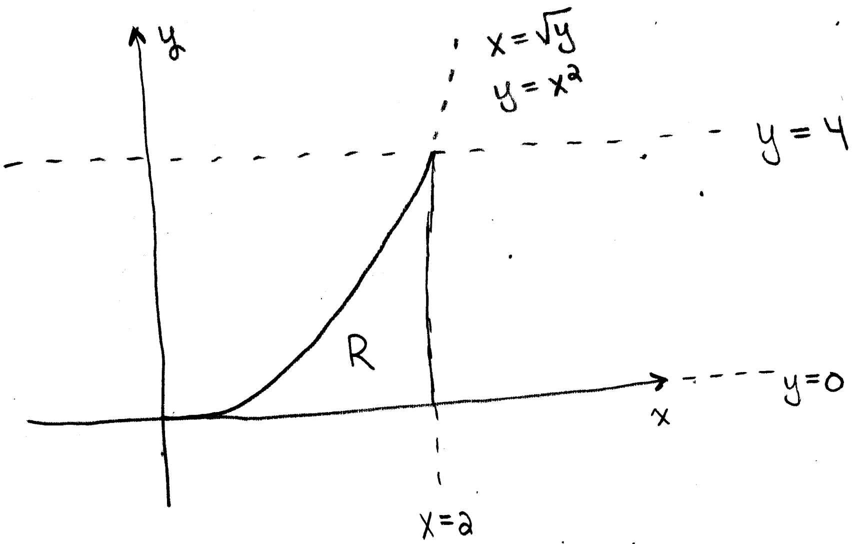
THERE IS A SINGLE CRITICAL POINT: $(2,-1)$

$d(2,-1) > 0$ AND $f_{xx}(2,-1) > 0$

$\Downarrow$

$f(2,-1) = -3$ IS A RELATIVE MIN.

8. (10 points) Consider the iterated integral $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$. Sketch the region of integration, reverse the order of integration, and evaluate.



$$\int_{x=0}^{x=2} \int_{y=0}^{y=x^2} e^{x^3}\, dy\, dx = \int_0^2 y e^{x^3} \Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^2 x^2 e^{x^3}\, dx = \frac{1}{3}\int_{u=0}^{u=8} e^u\, du = \frac{1}{3} e^u \Big|_0^8 = \boxed{\frac{1}{3}e^8 - \frac{1}{3} \approx 993.32}$$

$u = x^3$
$du = 3x^2\, dx$

9. (6 points) Assume that the equation $x^3 + y^2x - 3 = 0$ implicitly defines y as a function of x. Use partial derivatives to find dy/dx.

$$F(x,y) = x^3 + y^2x - 3$$

$$\boxed{\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(3x^2+y^2)}{2xy}}$$