Math 233 - Test 3b
April 14, 2022

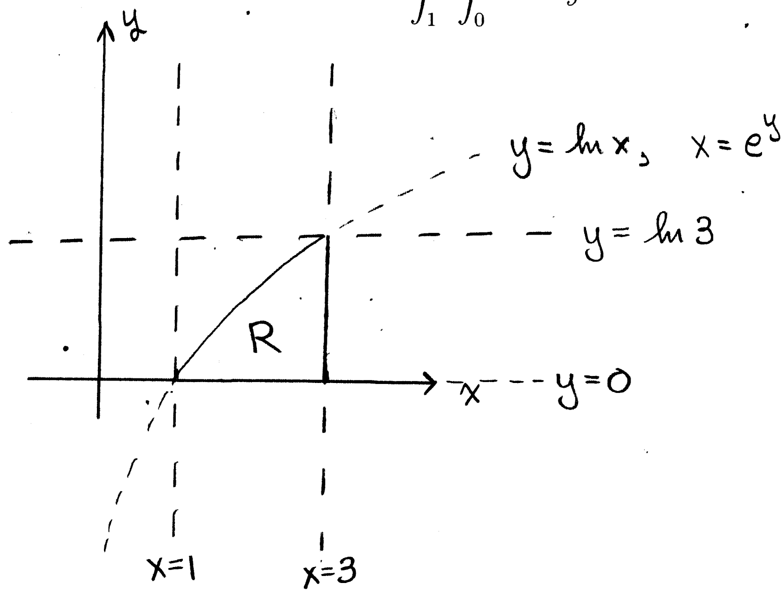
Name key

Score

Show all work to receive full credit. Supply explanations where necessary. You must work individually on this test. This test is due April 19.

1. (10 points) Sketch the region of integration, reverse the order of integration, and evaluate.

$$\int_1^3 \int_0^{\ln x} x\,dy\,dx$$



$$\int_{y=0}^{y=\ln 3} \int_{x=e^y}^{x=3} x\,dx\,dy = \int_0^{\ln 3} \frac{1}{2}x^2 \Big|_{e^y}^{3}\,dy$$

$$= \int_0^{\ln 3} \left(\frac{9}{2} - \frac{1}{2}e^{2y}\right) dy = \frac{9}{2}y - \frac{1}{4}e^{2y}\Big|_0^{\ln 3}$$

$$= \left(\frac{9}{2}\ln 3 - \frac{9}{4}\right) - \left(0 - \frac{1}{4}\right)$$

$$= \boxed{\frac{9\ln 3}{2} - 2 \approx 2.94}$$

2. (15 points) Find and classify the critical points of $f(x, y) = 2y^2x - yx^2 + 4yx$.

$f_x(x,y) = 2y^2 - 2yx + 4y = 0 \Rightarrow 2y(y - x + 2) = 0$

$f_y(x,y) = 4yx - x^2 + 4x$

$y = 0$ OR $y = x - 2$

$y = 0$:

$-x^2 + 4x = 0$

$-x(x-4) = 0$

$x = 0,\ x = 4$

$y = x - 2$:

$4x^2 - 8x - x^2 + 4x = 0$

$3x^2 - 4x = 0$

$x(3x-4) = 0$

$x = 0,\ x = \frac{4}{3}$

$y = -2,\ y = -\frac{2}{3}$

FOUR CRIT. PTS:

$(0,0),\ (4,0),\ (0,-2),\ \left(\frac{4}{3}, -\frac{2}{3}\right)$

$$D(x,y) = \begin{vmatrix} -2y & 4y - 2x + 4 \\ 4y - 2x + 4 & 4x \end{vmatrix} = -8xy - (4y - 2x + 4)^2$$

$(0,0)$: $D(0,0) = -16$, $f(0,0) = 0$ $\Rightarrow$ $\boxed{(0,0,0) \text{ IS A SADDLE PT.}}$

$(4,0)$: $D(4,0) = -16$, $f(4,0) = 0$ $\Rightarrow$ $\boxed{(4,0,0) \text{ IS A SADDLE PT.}}$

$(0,-2)$: $D(0,-2) = -16$, $f(0,-2) = 0$ $\Rightarrow$ $\boxed{(0,-2,0) \text{ IS A SADDLE PT}}$

$\left(\frac{4}{3}, -\frac{2}{3}\right)$: $D\left(\frac{4}{3}, -\frac{2}{3}\right) = \frac{16}{3}$, $f_{xx}\left(\frac{4}{3}, -\frac{2}{3}\right) = \frac{4}{3}$ $\Rightarrow$ $f\left(\frac{4}{3}, -\frac{2}{3}\right) = -\frac{32}{27}$ IS A REL MIN.

3. (6 points) Consider the surface defined by the equation $z^2 = 2x^2 + 4x - 6y^2 - 2y + 1$. Find a set of parametric equations for the line normal to the surface at the point $(2, 1, -3)$.

$$F(x,y,z) = 2x^2 + 4x - 6y^2 - 2y + 1 - z^2$$

$F(2,1,-3) = 0 \Rightarrow$ OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$.

$$\vec{\nabla} F(x,y,z) = (4x+4)\hat{\imath} + (-12y-2)\hat{\jmath} + (-2z)\hat{k}$$

$$\vec{n} = \vec{\nabla} F(2,1,-3) = 12\hat{\imath} - 14\hat{\jmath} + 6\hat{k}$$

NORMAL LINE:

$$\boxed{\begin{aligned} x &= 12t + 2 \\ y &= -14t + 1 \\ z &= 6t - 3 \end{aligned}}$$

4. (3 points) Suppose that z is a differentiable function of s, t, u, and v. Also suppose that s, t, u, and v are differentiable functions of x and y. Write the chain rule formula for $\partial z/\partial x$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

5. (6 points) Suppose that you were standing at the point $(-1, 2, 39)$ on the graph of $f(x, y) = x^2 - 3xy + 4y^3$. If you face the direction that makes a $-45°$ angle with the positive x-axis, would you be looking uphill or downhill? At what slope?

$$\vec{u} = \cos(-45°)\,\hat{i} + \sin(-45°)\,\hat{j}$$

$$= \frac{\sqrt{2}}{2}\,\hat{i} - \frac{\sqrt{2}}{2}\,\hat{j}.$$

$$D_{\vec{u}} f(-1,2) = ?$$

$$\vec{\nabla} f(x,y) = (2x - 3y)\hat{i} + (-3x + 12y^2)\hat{j}$$

$$\vec{\nabla} f(-1,2) = -8\hat{i} + 51\hat{j}$$

$$D_{\vec{u}} f(-1,2) = \vec{\nabla} f(-1,2) \cdot \vec{u} = (-8)\left(\frac{\sqrt{2}}{2}\right) + (51)\left(-\frac{\sqrt{2}}{2}\right)$$

$$= \frac{-59\sqrt{2}}{2}$$

$\boxed{\text{Downhill with slope } -\frac{59\sqrt{2}}{2}.}$