# Math 233 - Final Exam B

May 12, 2022

Name key

Score ____________

**Show all work to receive full credit.** Each problem is worth 5 points–up to 2 points for the answer and up to 3 points for the supporting work or explanation. Place your final answer in the box provided.

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1. A particle is moved from the point $A(0,2,1)$ to the point $B(3,1,6)$ by applying the force $\vec{F} = 4\hat{\imath} + \hat{\jmath} + 3\hat{k}$. Find the projection of $\vec{F}$ onto $\vec{AB}$.

$$\vec{AB} = 3\hat{\imath} - \hat{\jmath} + 5\hat{k}$$

$$\mathrm{proj}_{\vec{AB}} \vec{F} = \frac{\vec{F}\cdot\vec{AB}}{\vec{AB}\cdot\vec{AB}} \vec{AB} = \frac{12-1+15}{9+1+25} \vec{AB} = \frac{26}{35} \vec{AB}$$

$$\boxed{\frac{26}{35}\left(3\hat{\imath} - \hat{\jmath} + 5\hat{k}\right)}$$

2. Find the volume of the parallelepiped determined by the vectors $\vec{u} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$, $\vec{v} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, and $\vec{w} = \hat{\jmath} - 2\hat{k}$.

$$\text{Signed Volume} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = (1)(-4-1) - (-1)(-2-0) + (2)(1-0)$$

$$= -5 - 2 + 2 = -5$$

$$\text{Volume} = |-5| = 5$$

$$\boxed{5}$$

3. The line $L_1$ has symmetric equations

$$\frac{x-3}{2} = \frac{2-y}{3} = \frac{z}{7}.$$

The line $L_2$ is parallel to $L_1$ and passes through the point $(1, 2, -3)$. Find a set of parametric equations for $L_2$.

$\vec{v}$ = DIRECTION OF $L_1$ $= 2\hat{i} - 3\hat{j} + 7\hat{k}$

POINT $(1, 2, -3)$

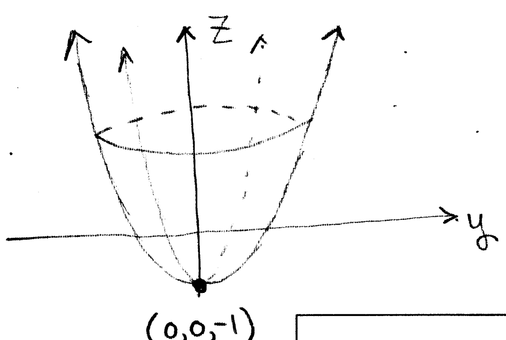
$$\boxed{x = 1 + 2t, \quad y = 2 - 3t, \quad z = -3 + 7t}$$

4. Let $G(x, y, z) = \sqrt{x^2 + y^2 - z}$. Describe the level ~~curve~~ SURFACE $G(x, y, z) = 1$.

$$\sqrt{x^2 + y^2 - z} = 1$$

$$\Rightarrow x^2 + y^2 - z = 1$$

$$\Rightarrow z = x^2 + y^2 - 1$$




CIRCULAR PARABOLOID WITH VERTEX AT (0,0,-1) OPENING UP THE Z-AXIS.

5. Find the limit or show that it does not exist: $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ %  More work!

$$\lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x\,\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})}{\cancel{(\sqrt{x}-\sqrt{y})}}$$

$$= \lim_{(x,y)\to(0,0)} \left[x(\sqrt{x}+\sqrt{y})\right] = 0$$

$$\boxed{0}$$

6. The temperature at a point in a solid is given by

$$T(x, y, z) = \frac{xyz}{1 + x^2 + y^2 + z^2}.$$

Use differentials to estimate the change in temperature from the point $(1, 1, 1)$ to the point $(1.05, -0.98, 1.02)$.

$$T_x(x,y,z) = \frac{(1+x^2+y^2+z^2)(yz) - (xyz)(2x)}{(1+x^2+y^2+z^2)^2} \qquad T_x(1,1,1) = \frac{2}{16} = \frac{1}{8}$$

$T_y(1,1,1) = T_z(1,1,1) = \frac{1}{8}$ (SYMMETRY IN ROLES OF VARIABLES!)

$$\Delta T \approx \frac{1}{8}\Delta x + \frac{1}{8}\Delta y + \frac{1}{8}\Delta z = \frac{1}{8}(0.05 - 1.98 + 0.02)$$

$$= \frac{-1.91}{8} = -0.23875$$

$$\boxed{\Delta T \approx \frac{-1.91}{8} = -0.23875}$$

7. Find an equation of the plane tangent to graph of $\sin(xz) - 4\cos(yz) = 4$ at the point $(\pi, \pi, 1)$.

$F(x,y,z) = \sin(xz) - 4\cos(yz)$

OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 4$

$\vec{\nabla} F(x,y,z) = z\cos(xz)\hat{\imath} + 4z\sin(yz)\hat{\jmath} + \left(x\cos(xz) + 4y\sin(yz)\right)\hat{k}$

$\vec{n} = \vec{\nabla} F(\pi, \pi, 1) = -\hat{\imath} + 0\hat{\jmath} + (-\pi)\hat{k}$

TANGENT PLANE : $-(x-\pi) + (-\pi)(z-1) = 0$

$-x + \pi - \pi z + \pi = 0$

$$\boxed{x + \pi z = 2\pi}$$

8. On a certain mountain, the elevation $z$ above the point $(x, y)$ is given by

$$z = 2000 - 2x^2 - 4y^2,$$

where the coordinates are measured in meters. Assume that the positive $x$-axis points east, and the positive $y$-axis points north. A climber at the point $(-20, 5, 1100)$ uses a compass reading to walk northeast. Will the climber ascend or descend? At what rate?

NORTHEAST $\Rightarrow$ $\vec{u} = \cos 45^\circ \hat{\imath} + \sin 45^\circ \hat{\jmath}$

$= \frac{\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath}$

$\vec{\nabla} z = -4x\hat{\imath} - 8y\hat{\jmath}$

$\vec{\nabla} z \Big|_{(-20,5)} = 80\hat{\imath} - 40\hat{\jmath}$

$\vec{\nabla} z \cdot \vec{u} = 40\sqrt{2} - 20\sqrt{2}$

$= 20\sqrt{2} \approx 28.28$ POSITIVE ... ASCEND

$$\boxed{\text{ASCEND. RATE} = 20\sqrt{2} \approx 28.28}$$

9. Find the critical point(s) of $f(x, y) = 3x^2 - 2xy + y^2 - 8y$. Then use the second partials test to classify your critical point(s).

$$f_x(x,y) = 6x - 2y = 0$$
$$f_y(x,y) = -2x + 2y - 8 = 0$$
$$4x - 8 = 0.$$
$$x = 2$$
$$y = 6$$

$$D = \begin{vmatrix} 6 & -2 \\ -2 & 2 \end{vmatrix} = 12 - 4 = 8$$

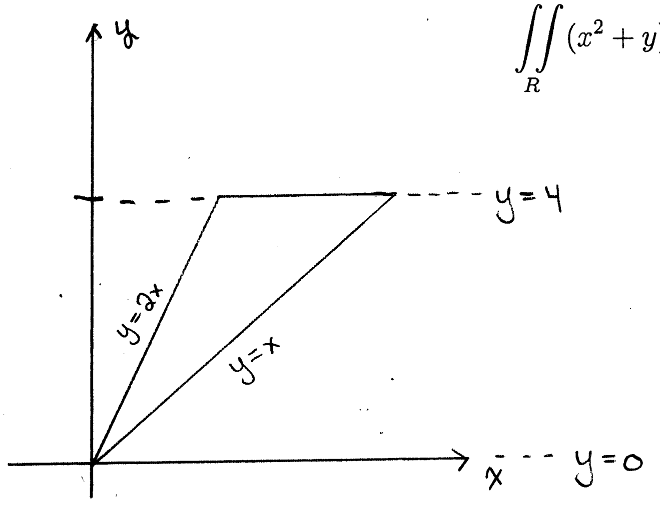
$D(2,6) > 0$ AND $f_{xx}(2,6) > 0$

$\Rightarrow f(2,6) = -24$ IS A REL. MIN.

$$\boxed{f(2,6) = -24 \text{ IS A REL. MIN.}}$$

10. Let $R$ be the region in the first quadrant bounded by graphs of $y = 2x$; $y = x$, and $y = 4$. Write the double integral as an iterated integral and evaluate.

$$\iint_R (x^2 + y)\, dA$$




$$\int_{y=0}^{y=4} \int_{x=\frac{y}{2}}^{x=y} (x^2 + y)\, dx\, dy$$

$$= \int_0^4 \left(\frac{y^3}{3} + y^2\right) - \left(\frac{y^3}{24} + \frac{y^2}{2}\right) dy$$

$$= \int_0^4 \left(\frac{7y^3}{24} + \frac{y^2}{2}\right) dy$$

$$\boxed{\frac{88}{3} = 29.\overline{3}}$$

$$\frac{7}{96}y^4 + \frac{y^3}{6}\Big|_0^4 = \frac{1792}{96} + \frac{64}{6} = \frac{2816}{96} = \frac{88}{3}$$

$x^2 + y^2 = 2x$

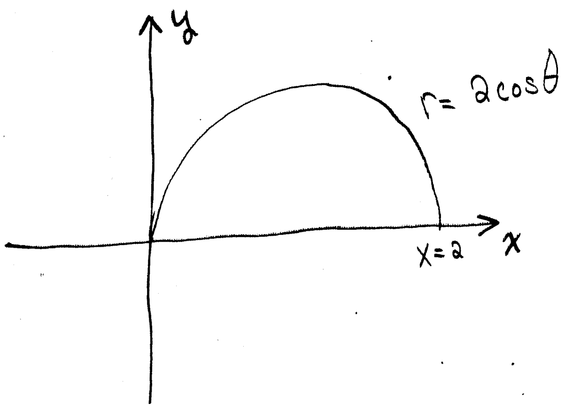
$\Rightarrow r^2 = 2r\cos\theta$

$r = 2\cos\theta$

11. The first octant space region under the cone $z = \sqrt{x^2+y^2}$ and above the circle $y^2 = 2x - x^2$ has volume given by

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$

Convert this integral to an iterated integral in polar coordinates. Do not evaluate.




$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2\cos\theta} r^2\, dr\, d\theta$$

$$\boxed{\int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta}$$

12. Evaluate the line integral $\int_C \vec{F}(x,y) \cdot d\vec{r}$, where $\vec{F}(x,y) = -y\hat{\imath} + x\hat{\jmath}$ and $C$ is the path along $y^2 = 3x$ from $(3,3)$ to $(0,0)$.

$$\int_C -y\,dx + x\,dy \qquad y^2 = 3x \Rightarrow 2y\,dy = 3\,dx$$

$$= \int_{y=3}^{y=0} -y\left(\frac{2y}{3}\right)dy + \frac{y^2}{3}\,dy = \int_3^0 -\frac{y^2}{3}\,dy = \int_0^3 \frac{y^2}{3}\,dy$$

$$= \left.\frac{y^3}{9}\right|_0^3$$

$$= 3$$

$$\boxed{3}$$