Math 233 - Quiz 12

May 4, 2023

Name key

Score _____

Show all work to receive full credit. Supply explanations when necessary. This quiz is due May 9.

1. (5 points) Use a double integral to find the area of the 1st-quadrant region inside both circles $r = 3\sin\theta$ and $r = \sqrt{3}\cos\theta$.

See graph on attached sheet.

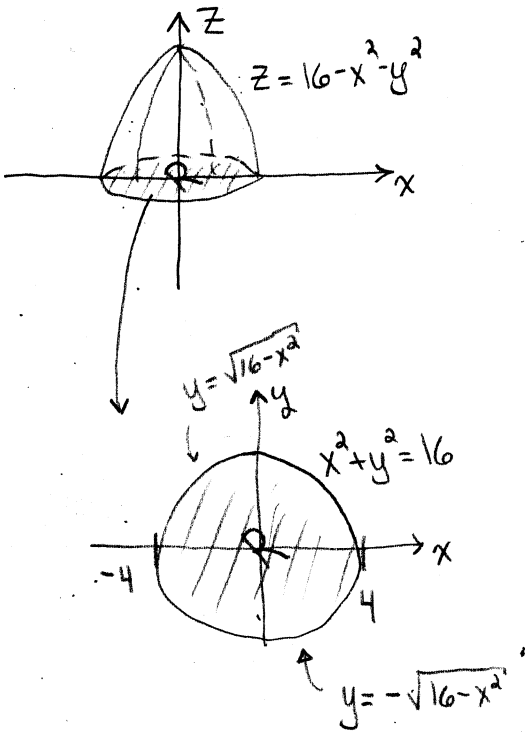
$$\text{Area} = \int_{r=0}^{r=3/2}\int_{\theta=\sin^{-1}\left(\frac{r}{3}\right)}^{\theta=\cos^{-1}\left(\frac{r}{\sqrt{3}}\right)} r\,d\theta\,dr = \int_{\theta=0}^{\theta=\frac{\pi}{6}}\int_{r=0}^{r=3\sin\theta} r\,dr\,d\theta + \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=\sqrt{3}\cos\theta} r\,dr\,d\theta$$

$$\int_0^{\pi/6}\frac{9}{2}\sin^2\theta\,d\theta + \int_{\pi/6}^{\pi/2}\frac{3}{2}\cos^2\theta\,d\theta = \frac{9}{2}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6} + \frac{3}{2}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

$\frac{1}{2} - \frac{1}{2}\cos 2\theta$ (Power-reducing) $\qquad \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$= \left[\frac{9\pi}{24} - \frac{9\sqrt{3}}{16} - 0 + 0\right] + \left[\frac{3\pi}{8} + 0 - \frac{3\pi}{24} - \frac{3\sqrt{3}}{16}\right] = \frac{5\pi}{8} - \frac{3\sqrt{3}}{4}$$

2. (5 points) Let S be the space region above the xy-plane and under the paraboloid $z = 16 - x^2 - y^2$. Set up the triple integral(s) necessary to compute the average value of $f(x, y, z) = 1 + x^2 + y^2 + z^3$ over S. Use technology to compute the average value.

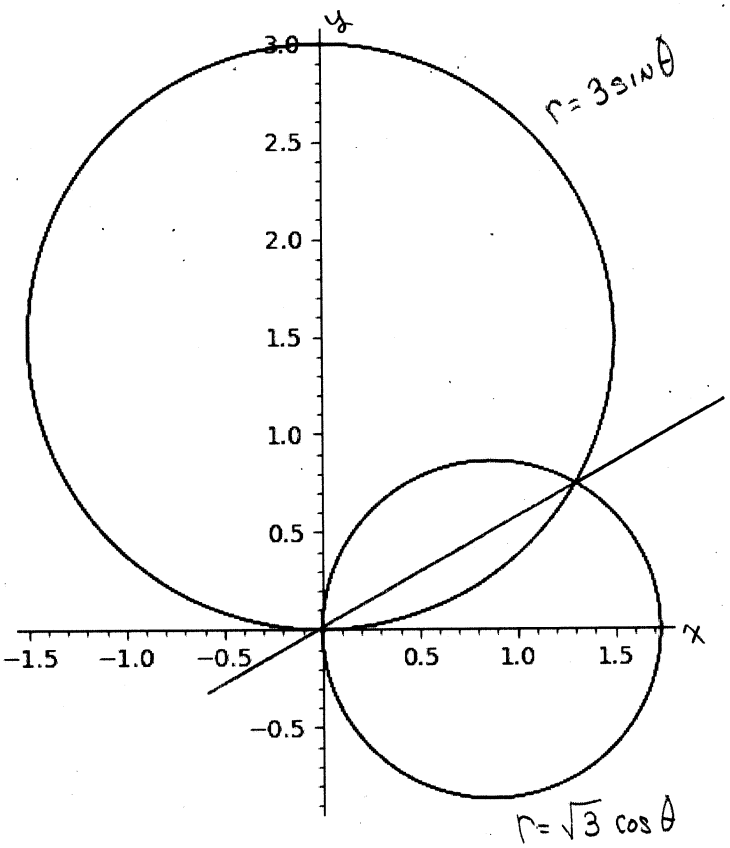


$$\text{Volume} = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{0}^{16-x^2-y^2} dz\,dy\,dx =$$

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4}\int_{z=0}^{z=16-r^2} r\,dz\,dr\,d\theta = 128\pi$$

$$\text{Average value} = \frac{1}{128\pi}\int_0^{2\pi}\int_0^4\int_0^{16-r^2}(1 + r^2 + z^3)\,r\,dz\,dr\,d\theta$$

$$= \frac{1}{128\pi}\left(\frac{798592\pi}{15}\right) = \frac{6239}{15}$$



$3\sin\theta = \sqrt{3}\cos\theta$

$\tan\theta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} = \frac{1/2}{\sqrt{3}/2}$

$\theta = \frac{\pi}{6}$

$3\sin\frac{\pi}{6} = \frac{3}{2}$