# Math 233 - Test 1

February 9, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) The vector $\vec{w}$ lies in the $xy$-plane, has magnitude 8, and makes a 120° angle with the positive $x$-axis. Find the projection of $\vec{w}$ onto $\vec{v} = -10\hat{\imath} + 2\hat{\jmath}$.

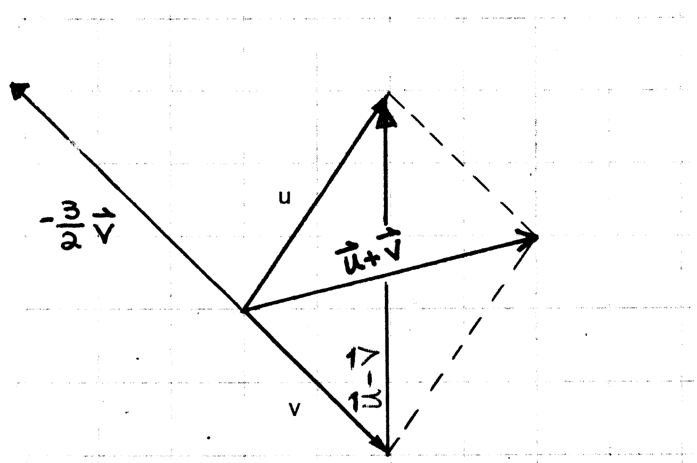
$$\vec{w} = 8 \cos 120^\circ \hat{i} + 8 \sin 120^\circ \hat{j}$$

$$\vec{w} = -4\hat{i} + 4\sqrt{3}\hat{j}$$

$$\text{proj}_{\vec{v}} \vec{w} = \frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{40+8\sqrt{3}}{104}\vec{v} = \boxed{\left(\frac{5+\sqrt{3}}{13}\right)\left(-10\hat{i}+2\hat{j}\right)}$$

$$\approx \langle -5.1785, 1.0357 \rangle$$

2. (6 points) The figure below shows the vectors $\vec{u}$ and $\vec{v}$. Sketch and label the vectors $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, and $-\frac{3}{2}\vec{v}$.



3. (6 points) Show that the points $R(14, 39, 3)$, $S(5, -3, 0)$, and $T(-1, -31, -2)$ are collinear.

$\overrightarrow{RS} = -9\hat{\imath} - 42\hat{\jmath} - 3\hat{k}$

$\overrightarrow{RT} = -15\hat{\imath} - 70\hat{\jmath} - 5\hat{k}$

$\boxed{\overrightarrow{RS} = \frac{3}{5}\overrightarrow{RT}}$

$\overrightarrow{RS}$ AND $\overrightarrow{RT}$ ARE PARALLEL

AND SHARE THEIR INITIAL PT.

$\Rightarrow$ R, S, T ARE COLLINEAR.

4. (6 points) Determine the measure of the angle that $\vec{w} = \sqrt{2}\,\hat{\imath} + 3\hat{\jmath} - \sqrt{7}\,\hat{k}$ makes with the positive $y$-axis. Write your answer in degrees.

$$\cos\theta = \frac{\vec{\omega}\cdot\hat{\jmath}}{\|\vec{\omega}\|\,\|\hat{\jmath}\|} = \frac{3}{\sqrt{2+9+7}} = \frac{3}{\sqrt{18}} = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = \frac{\pi}{4} = 45^\circ}$$

5. (9 points) Let $\vec{v} = \hat{\imath} + \hat{\jmath} - 7\hat{k}$ and $\vec{w} = -5\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$. Show that $\vec{v} \times \vec{w}$ is orthogonal to $2\vec{v} + \vec{w}$.

$$2\vec{v} + \vec{\omega} = -3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$$

$$\vec{v} \times \vec{\omega} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -7 \\ -5 & 2 & 9 \end{vmatrix} = \hat{\imath}(9+14) - \hat{\jmath}(9-35) + \hat{k}(2+5)$$

$$= 23\hat{\imath} + 26\hat{\jmath} + 7\hat{k}$$

$$(2\vec{v} + \vec{\omega}) \cdot (\vec{v} \times \vec{\omega}) = -3(23) + 4(26) - 5(7)$$

$$= -69 + 104 - 35 = 0 \checkmark$$

6. (9 points) Consider the line in space that passes through the points $P(5,-2,1)$ and $Q(-6,3,-5)$.

(a) Find symmetric equations for the line.

$\vec{PQ} = -11\hat{i} + 5\hat{j} - 6\hat{k}$

Using P, Symmetric equations : $\boxed{\frac{x-5}{-11} = \frac{y+2}{5} = \frac{z-1}{-6}}$

(b) Find a set of parametric equations for the segment $\overline{PQ}$.

$$\boxed{\begin{aligned} x &= -11t + 5 \\ y &= 5t - 2 \qquad 0 \le t \le 1 \\ z &= -6t + 1 \end{aligned}}$$

P coincides w/ $t=0$

Q coincides w/ $t=1$

(c) Find the midpoint of the segment $\overline{PQ}$. Referring to part (b), what value of your parameter coincides the midpoint?

Midpoint $= \left(\frac{5+-6}{2}, \frac{-2+3}{2}, \frac{1+-5}{2}\right)$

$= \boxed{\left(-\frac{1}{2}, \frac{1}{2}, -2\right)}$ Coincides w/ $t = \frac{1}{2}$

7. (10 points) Find an equation of the plane that contains the point $P(2,4,-1)$ and all points on the line $\ell$. Symmetric equations for $\ell$ are shown below.

$$\text{Line } \ell: \quad \frac{x-1}{2} = y+4 = \frac{z-5}{2}$$

Q = Point of $\ell$ = $(1,-4,5)$

$\vec{PQ} = -\hat{i} - 8\hat{j} + 6\hat{k}$

Direction of $\ell = \vec{v} = 2\hat{i} + \hat{j} + 2\hat{k}$

$\vec{n} = \vec{PQ} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -8 & 6 \\ 2 & 1 & 2 \end{vmatrix}$

$= \hat{i}(-16-6) - \hat{j}(-2-12) + \hat{k}(-1+16)$

$\vec{n} = -22\hat{i} + 14\hat{j} + 15\hat{k}$

Using P...

$-22(2) + 14(4) + 15(-1)$

$= -3$

$\boxed{-22x + 14y + 15z = -3}$

8. (10 points) If $P$ is a point on the line that has the direction of $\vec{v}$, then the distance from the line to a point $Q$ is given by

$$D = \frac{\|\vec{PQ} \times \vec{v}\|}{\|\vec{v}\|}.$$

Find the distance from $\overbrace{\text{the origin}}^{Q}$ to the line with parametric equations

$$x = 1 + t, \quad y = 3 + t, \quad z = 5 + 4t.$$

$P(1, 3, 5)$

$Q(0, 0, 0)$

$\vec{PQ} = -\hat{i} - 3\hat{j} - 5\hat{k}$

$\vec{v} = \hat{i} + \hat{j} + 4\hat{k}$

$$\vec{PQ} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -3 & -5 \\ 1 & 1 & 4 \end{vmatrix}$$

$= \hat{i}(-12+5) - \hat{j}(-4+5) + \hat{k}(-1+3)$

$= -7\hat{i} - \hat{j} + 2\hat{k}$

$\|\vec{PQ} \times \vec{v}\| = \sqrt{49+1+4} = \sqrt{54}$.

$\|\vec{v}\| = \sqrt{1+1+16} = \sqrt{18}$

$D = \dfrac{\sqrt{54}}{\sqrt{18}} = \boxed{\sqrt{3}}$

9. (6 points) Determine the measure of the angle between the planes. Write your answer in degrees rounded to the nearest integer.

$$x - 3y + 6z = 4, \qquad 5x + y - z = 4$$

$\vec{n}_1 = \hat{i} - 3\hat{j} + 6\hat{k}$

$\vec{n}_2 = 5\hat{i} + \hat{j} - \hat{k}$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \, \|\vec{n}_2\|} = \frac{|5-3-6|}{\sqrt{46}\sqrt{27}} = \frac{4}{\sqrt{46}\sqrt{27}}$$

$\theta = 83.4828\ldots^\circ$

$\Rightarrow \boxed{\theta \approx 83^\circ}$

10. (8 points) Consider the vector-valued function $\vec{r}(t) = 2t\,\hat{\imath} - t^2\,\hat{\jmath}$.

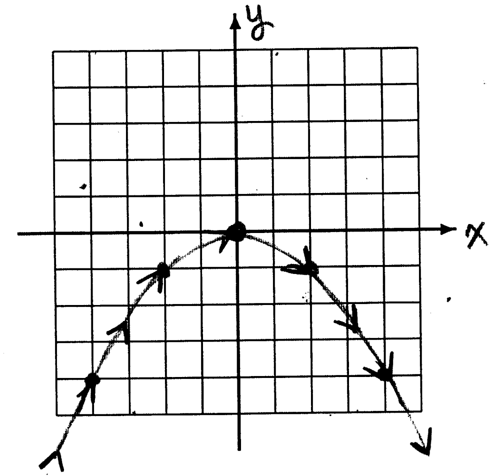
(a) Sketch the graph of $\vec{r}(t)$. Show or describe the orientation of the curve.

$x = 2t \rightarrow t = \frac{x}{2}$

$y = -t^2$

$\Downarrow$

$y = \frac{-x^2}{4}$




ORIENTATION: DIRECTION OF INCREASING $t$ IS DIRECTION OF INCREASING $x$. (SEE ARROWS.)

(b) Compute $\|\vec{r}(t)\|$.

$\|\vec{r}(t)\| = \sqrt{(2t)^2 + (-t^2)^2} \Rightarrow \boxed{\|\vec{r}(t)\| = \sqrt{4t^2 + t^4}}$

$= |t|\sqrt{4+t^2}$

(c) Find a (nonzero) vector-valued function that is orthogonal to $\vec{r}(t)$ for every real number $t$.

$\vec{r}$ IS A 2D VECTOR. THE "REVERSE AND SWITCH A SIGN" STRATEGY WILL WORK.

$\boxed{\vec{r}_{\perp}(t) = t^2\,\hat{\imath} + 2t\,\hat{\jmath}}$

(d) Describe the graph of the vector-valued function $\vec{r}(t) = 2t\,\hat{\imath} - t^2\,\hat{\jmath} + t\,\hat{k}$.

TAKE THE PARABOLA ABOVE (I.e., THE GRAPH OF $\vec{r}(t) = 2t\hat{\imath} - t^2\hat{\jmath}$) AND TILT IT ABOUT THE $y$-AXIS SO THE RIGHT SIDE COMES OUT OF THE PAPER AND THE LEFT SIDE GOES INTO THE PAPER. THE GRAPH IS THE SAME PARABOLA, BUT IN THE $X = 2Z$ PLANE.

11. (8 points) Let $\vec{r}(t) = \dfrac{e^t - 1}{t}\,\hat{\imath} + \dfrac{\sin t}{t}\,\hat{\jmath} + \sin(\pi t)\,\hat{k}$.

(a) Determine the domain of $\vec{r}$.

$$\text{Domain} = \{ t \in \mathbb{R} : t \neq 0 \}$$

(b) Compute $\vec{r}(4)$.

$$\vec{r}(4) = \left(\frac{e^4 - 1}{4}\right)\hat{\imath} + \left(\frac{\sin 4}{4}\right)\hat{\jmath} + \sin(4\pi)\,\hat{k}$$

$$\approx \boxed{13.40\,\hat{\imath} - 0.19\,\hat{\jmath} + 0\,\hat{k}}$$

(c) Compute $\lim\limits_{t\to 1} \vec{r}(t)$.

$\vec{r}$ is continuous at $t = 1$

$$\lim_{t\to 1} \vec{r}(t) = \vec{r}(1) = (e-1)\,\hat{\imath} + \sin(1)\,\hat{\jmath} + \sin \pi\,\hat{k}$$

$$\approx \boxed{1.72\,\hat{\imath} + 0.84\,\hat{\jmath} + 0\,\hat{k}}$$

(d) Compute $\lim\limits_{t\to 0} \vec{r}(t)$.

$$\underbrace{\left(\lim_{t\to 0} \frac{e^t - 1}{t}\right)}_{}\hat{\imath} + \underbrace{\left(\lim_{t\to 0} \frac{\sin t}{t}\right)}_{1}\hat{\jmath} + \underbrace{\left(\lim_{t\to 0} \sin \pi t\right)}_{0}\hat{k}$$

L.H. Rule $\lim\limits_{t\to 0} \dfrac{e^t}{1} = 1$

$$= \boxed{\hat{\imath} + \hat{\jmath}}$$

The following problem makes up the take-home portion of the test. This portion of the test is due February 14, 2023. You must work on your own.

12. (14 points) Consider the following planes

$$P_1 : 2x - 3y + 8z = 10, \qquad P_2 : x + 2y + 4z = 4.$$

(a) Show that the planes are not parallel.

$\vec{n}_1$ = NORMAL VECTOR FOR $P_1$

$= 2\hat{i} - 3\hat{j} + 8\hat{k}$

$\vec{n}_2$ = NORMAL VECTOR FOR $P_2$

$= \hat{i} + 2\hat{j} + 4\hat{k}$

$\vec{n}_1$ IS NOT A SCALAR MULTIPLE OF $\vec{n}_2$.

NORMALS NOT PARALLEL

$\Rightarrow$ PLANES NOT PARALLEL.

(b) Find a set of parametric equations for the line of intersection of the planes.

DIRECTION OF LINE IS

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 8 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= \hat{i}(-12-16) - \hat{j}(8-8) + \hat{k}(4+3) = -28\hat{i} + 7\hat{k}$$

I'LL USE $4\hat{i} - \hat{k}$.

POINT ON LINE $\left(\frac{32}{7}, \frac{-2}{7}, 0\right)$

(SEE BACK)

$$\boxed{x = 4t + \frac{32}{7}, \quad y = -\frac{2}{7}, \quad z = -t}$$

THERE ARE LOTS OF SOLUTIONS.

(c) Find the distance from the point $R(9, 1, -3)$ to the plane $P_1$.

$$d = \frac{|2(9) - 3(1) + 8(-3) - 10|}{\sqrt{4 + 9 + 64}} = \boxed{\frac{19}{\sqrt{77}} \approx 2.165}$$

(d) Find symmetric equations for the line through $(2, 3, 4)$ and normal to the plane $P_2$.

$\vec{n}_2 = \hat{i} + 2\hat{j} + 4\hat{k}$.

POINT $(2, 3, 4)$

$$\frac{x-2}{1} = \frac{y-3}{2} = \frac{z-4}{4}$$

OR

$$\boxed{x - 2 = \frac{y-3}{2} = \frac{z-4}{4}}$$

$$2x - 3y + 8z = 10$$
$$x + 2y + 4z = 4$$

$$z = 0 \Rightarrow \begin{array}{l} 2x - 3y = 10 \\ x + 2y = 4 \end{array} \Rightarrow \begin{array}{r} 2x - 3y = 10 \\ -2x - 4y = -8 \\ \hline -7y = 2 \end{array}$$

$$y = -\frac{2}{7}$$

$$x + 2\left(-\frac{2}{7}\right) = 4$$

$$x = \frac{28}{7} + \frac{4}{7} = \frac{32}{7}$$

$$\left(\frac{32}{7}, -\frac{2}{7}, 0\right)$$