Math 233 - Test 2
March 9, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) For $t \geq 0$, let $\vec{r}(t) = (\cos t + t\sin t)\,\hat{\imath} + (\sin t - t\cos t)\,\hat{\jmath}$. Compute the principal unit tangent vector, $\hat{T}(t)$.

$$\vec{r}\,'(t) = (-\sin t + \sin t + t\cos t)\,\hat{\imath} + (\cos t - \cos t + t\sin t)\,\hat{\jmath}$$

$$= t\cos t\,\hat{\imath} + t\sin t\,\hat{\jmath}$$

$$\|\vec{r}\,'(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t, \quad t > 0$$

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$$

2. (8 points) The velocity vector of a moving particle is given by

$$\vec{v}(t) = (\cos t)\,\hat{\imath} + (5\sin t)\,\hat{\jmath} + e^{-t}\,\hat{k}.$$

Find the position vector if the particle's motion began (at $t = 0$) at the point $(2, 7, 4)$.

INTEGRATE TO GET

$$\vec{r}(t) = (\sin t + c_1)\,\hat{\imath} + (-5\cos t + c_2)\,\hat{\jmath} + (-e^{-t} + c_3)\,\hat{k}$$

$$\vec{r}(0) = 2\hat{\imath} + 7\hat{\jmath} + 4\hat{k} \Rightarrow c_1 = 2, \quad c_2 = 12, \quad c_3 = 5$$

$$\vec{r}(t) = (2 + \sin t)\,\hat{\imath} + (12 - 5\cos t)\,\hat{\jmath} + (5 - e^{-t})\,\hat{k}$$

3. (8 points) A curve in the xy-plane is described by the following parametric equations. Find the curvature function, $\kappa(t)$.

$$x = \frac{t^3}{3}, \quad y = \frac{t^2}{2}$$

$$\vec{r}(t) = \frac{t^3}{3}\hat{\imath} + \frac{t^2}{2}\hat{\jmath}$$

$$\vec{r}\,'(t) = t^2\,\hat{\imath} + t\,\hat{\jmath}$$

$$\vec{r}\,''(t) = 2t\,\hat{\imath} + \hat{\jmath}$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t^2 & t & 0 \\ 2t & 1 & 0 \end{vmatrix}$$

$$= (t^2 - 2t^2)\hat{k} = -t^2\hat{k}$$

$$\|\vec{r}\,' \times \vec{r}\,''\| = t^2$$

$$\|\vec{r}\,'\| = \sqrt{t^4 + t^2} = |t|\sqrt{t^2+1}$$

$$K(t) = \frac{t^2}{|t|^3 (t^2+1)^{3/2}}$$

$$K(t) = \frac{1}{|t|\,(t^2+1)^{3/2}}$$

4. (12 points) Let $\vec{r}(t) = t\,\hat{\imath} - \sin 4t\,\hat{\jmath} - \cos 4t\,\hat{k}$. Starting from $t = 0$, find the arc-length parameter, $s(t)$, and then reparameterize $\vec{r}$ in terms of s.

$$\vec{r}\,'(t) = \hat{\imath} - 4\cos 4t\,\hat{\jmath} + 4\sin 4t\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 16} = \sqrt{17}$$

$$S = \int_0^t \sqrt{17}\, d\tau = \sqrt{17}\, t$$

$$t = \frac{S}{\sqrt{17}}$$

$$\vec{R}(s) = \frac{s}{\sqrt{17}}\hat{\imath} - \sin\frac{4s}{\sqrt{17}}\hat{\jmath} - \cos\frac{4s}{\sqrt{17}}\hat{k}$$

Follow-up: Show that when the function is reparameterized, its derivative has magnitude 1.

$$\vec{R}\,'(s) = \frac{1}{\sqrt{17}}\hat{\imath} - \frac{4}{\sqrt{17}}\cos\frac{4s}{\sqrt{17}}\hat{\jmath} + \frac{4}{\sqrt{17}}\sin\frac{4s}{\sqrt{17}}\hat{k}$$

$$\|\vec{R}\,'(s)\| = \sqrt{\frac{1}{17} + \frac{16}{17}} = \sqrt{\frac{17}{17}} = 1$$

5. (10 points) Let $\vec{r}(t) = (2t+3)\,\hat{\imath} + (t^2-1)\,\hat{\jmath}$. Compute the tangential and normal components of acceleration.

$\vec{r}\,'(t) = 2\hat{\imath} + 2t\hat{\jmath}$

$\vec{r}\,''(t) = 2\hat{\jmath}$

$\|\vec{r}\,'(t)\| = \sqrt{4+4t^2}$

$= 2\sqrt{1+t^2}$

$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 4t \Rightarrow a_T = \frac{4t}{2\sqrt{1+t^2}} = \frac{2t}{\sqrt{1+t^2}}$

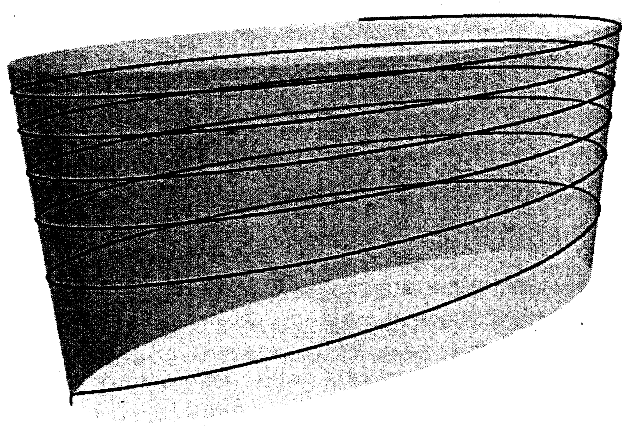
$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = 4\hat{k}$$

$$a_{\vec{N}} = \frac{4}{2\sqrt{1+t^2}} = \frac{2}{\sqrt{1+t^2}}$$

6. (8 points) A wire is wrapped around an elliptical steel tube so that the wire has the shape of the graph of

$$\vec{r}(t) = 6\cos(t)\,\hat{\imath} + 2\sin(t)\,\hat{\jmath} + \sqrt{t}\,\hat{k}, \quad 0 \le t \le 36,$$

where $\vec{r}$ is in centimeters. Set up the definite integral that gives the length of the wire. Use your calculator to approximate the value of your integral.



$\vec{r}\,'(t) = -6\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath} + \frac{1}{2}t^{-1/2}\hat{k}$

$$\text{ARC LENGTH} = \underbrace{\int_0^{36} \sqrt{36\sin^2 t + 4\cos^2 t + \frac{1}{4t}}\,dt}_{\text{ACTUALLY, AN IMPROPER INTEGRAL. LET LOWER BOUND APPROACH ZERO FROM THE RIGHT.}} \approx 153.3 \text{ cm}$$

7. (8 points) A baseball, hit 3 feet above the ground, leaves the bat at an angle of 45° and is caught by an outfielder at a height of 3 feet above the ground and 300 feet from home plate. What is the initial speed of the ball? (To receive full credit, you must write and use the vector-valued function $\vec{r}(t)$ that gives the position of the ball at time t. Also ignore air resistance and use $g \approx 32\,\text{ft}\,\text{s}^{-2}$.)

$$\vec{r}(t) = \frac{\sqrt{2}}{2} V_0 t\,\hat{i} + \left(-16t^2 + \frac{\sqrt{2}}{2} V_0 t + 3\right)\hat{j}$$

$\frac{\sqrt{2}}{2} V_0 t = 300$ WHEN $-16t^2 + \frac{\sqrt{2}}{2} V_0 t + 3 = 3$

$$-16t^2 + 300 = 0$$

$$t^2 = \frac{300}{16}$$

$$t = \frac{10\sqrt{3}}{4} = \frac{5\sqrt{3}}{2}$$

$$V_0 = \frac{300}{\frac{\sqrt{2}}{2} t} = \frac{300}{\frac{5\sqrt{6}}{4}} = \boxed{\frac{240}{\sqrt{6}}\ \text{FT/SEC}} \approx 97.98\ \text{FT/SEC}$$

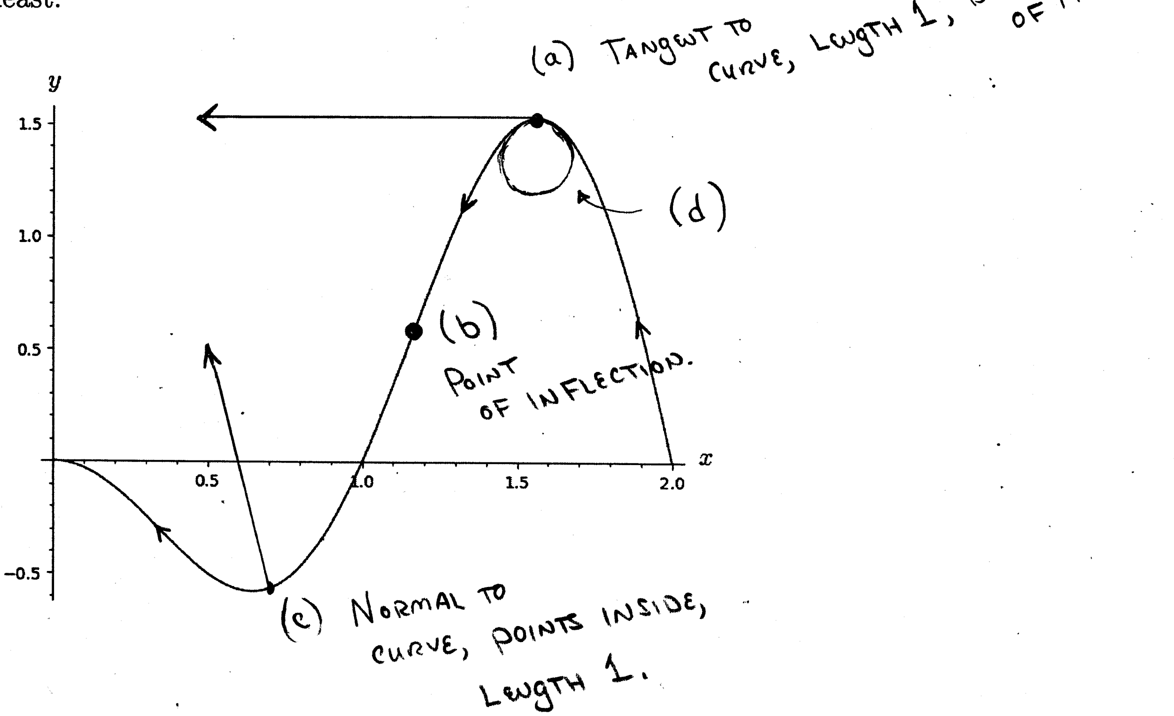
8. (6 points) Suppose a particle moves along the given curve from **right to left**. Sketch and label each of the following. Make note of the scale.

(a) The principal unit tangent vector at the point of greatest curvature.

(b) A point where the principal unit normal vector does not exist.

(c) The principal unit normal vector at the point where $x = 0.7$.

(d) (1 pt extra credit) The circle of curvature at the point where the radius of curvature is least.



9. (10 points) Consider the function $F(x,y) = \sqrt{1+x^2-|y|}$.

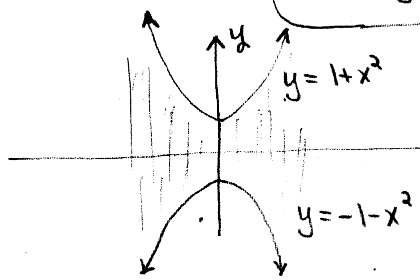
(a) What is the domain of F?

$$1+x^2-|y| \geq 0 \Rightarrow |y| \leq 1+x^2 \Rightarrow \{(x,y): -1-x^2 \leq y \leq 1+x^2\}$$

$$-1-x^2 \leq y \leq 1+x^2$$

(b) Sketch the domain in the xy-plane.

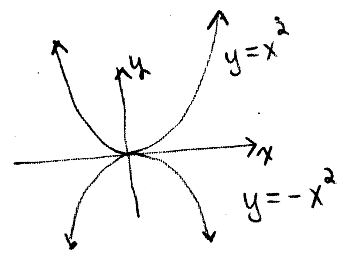
ALL POINTS ON & BETWEEN $y=1+x^2$ AND $y=-1-x^2$



(c) Sketch the level curve $F(x,y)=1$.

$$\sqrt{1+x^2-|y|} = 1 \Rightarrow x^2-|y| = 0$$

$$\Rightarrow |y| = x^2 \Rightarrow y = \pm x^2$$



(d) What is the range of F?

$\{z : z \geq 0\}$.

(e) Compute $F(4,-8)$.

$$F(4,-8) = \sqrt{1+16-8} = \sqrt{9} = \boxed{3}$$

10. (8 points) Identify the graph of the **surface in space** described by each equation.

(a) $\dfrac{y}{5} = 8x^2 + 7z^2$

LEVEL CURVES IN X, Y, Z ORDER:
PARABOLAS, ELLIPSES, PARABOLAS.

$\boxed{\text{PARABOLOID}}$

(b) $x^2 - \sqrt{8} = y^2 + z^2$

CIRCLES, HYPERBOLAS, HYPERBOLAS
IF X IS TOO SMALL, NO LEVEL CURVE.

HYPERBOLOID OF TWO SHEETS

(c) $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

MISSING X

ELLIPTICAL CYLINDER CENTERED ON X-AXIS

(d) $x^2 - y^2 + z^2 = 0$

HYPERBOLAS, CIRCLES, HYPERBOLAS

POINT AT $(0,0,0)$

$\boxed{\text{CONE}}$

11. (6 points) Let $G(x, y, z) = x^2 + y^2 + z^2$.

(a) Compute $G(-1, 2, -3)$.

$$G(-1,2,-3) = (-1)^2 + (2)^2 + (-3)^2 = 1+4+9 = \boxed{14}$$

(b) What are the domain and range of G?

Domain $= \mathbb{R}^3$

Range $= [0, \infty)$

(c) Describe, in detail, the level surface $G(x, y, z) = 9$.

$x^2+y^2+z^2=9 \rightarrow$ Sphere centered at $(0,0,0)$ with radius 3.

12. (10 points) Compute each limit.

(a) $\lim\limits_{(x,y)\to(3,3)} \dfrac{(2x+y)^2 - (5y^2+4xy)}{x-y}$ 0/0 More work

$$= \lim_{(x,y)\to(3,3)} \frac{4x^2+4xy+y^2-5y^2-4xy}{x-y} = \lim_{(x,y)\to(3,3)} \frac{4x^2-4y^2}{x-y}$$

$$= \lim_{(x,y)\to(3,3)} 4(x+y) = \boxed{24}$$

(b) $\lim\limits_{(x,y)\to(2,1)} \dfrac{xy^2 - 2y^2}{\sqrt{x}-\sqrt{2}}$ 0/0 More work

$$= \lim_{(x,y)\to(2,1)} \frac{y^2(x-2)}{\sqrt{x}-\sqrt{2}} \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$$

$$= \lim_{(x,y)\to(2,1)} \frac{y^2\cancel{(x-2)}(\sqrt{x}+\sqrt{2})}{\cancel{x-2}} = (1)^2(\sqrt{2}+\sqrt{2})$$

$$= \boxed{2\sqrt{2}}$$