Math 233 - Test 3

April 13, 2023

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary.

1. (14 points) Use the two-path test to show that each limit fails to exist.

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$

ALONG $X=0$: $\displaystyle \lim_{y\to 0} \frac{y^3}{y^2} = \lim_{y\to 0} y = 0$

ALONG $X=y$: $\displaystyle \lim_{y\to 0} \frac{y^2+y^3}{y^2+y^2} = \lim_{y\to 0} \frac{1+y}{2} = \frac{1}{2}$

TWO DIFFERENT LIMITS ALONG TWO PATHS. LIMIT DNE.

(b) $\displaystyle \lim_{(x,y)\to(1,0)} \frac{(x-1)y^2}{(x-1)^3+y^4}$

ALONG $y=0$: $\displaystyle \lim_{X\to 1} \frac{0}{(X-1)^3} = \lim_{X\to 1} 0 = 0$

ALONG $y=X-1$: $\displaystyle \lim_{y\to 0} \frac{y^3}{y^3+y^4} = \lim_{y\to 0} \frac{1}{1+y} = 1$

TWO DIFFERENT LIMITS ALONG TWO PATHS. LIMIT DNE.

2. (2 points) If f is continuous at (x_0, y_0), then $\displaystyle \lim_{(x,y)\to(x_0,y_0)} f(x,y) =$ $f(x_0,y_0)$.

3. (8 points) Let $f(x, y) = \dfrac{xy}{x-y}$. Evaluate f_x and f_y at the point $(2, -2)$.

$$f_x(x,y) = \frac{(x-y)(y) - (xy)(1)}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

$$f_x(2,-2) = \frac{-4}{16} = \boxed{-\frac{1}{4}}$$

$$f_y(x,y) = \frac{(x-y)(x) - (xy)(-1)}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

$$f_y(2,-2) = \frac{4}{16} = \boxed{\frac{1}{4}}$$

4. (8 points) Show that $z = e^x \sin y$ satisfies the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

$$\frac{\partial z}{\partial x} = e^x \sin y \qquad \frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y \qquad \frac{\partial^2 z}{\partial x^2} = -e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0 \quad \checkmark$$

5. (2 points) Suppose f is defined on an open region in $\mathbb{R}^2$. We would expect $f_{xy} = f_{yx}$ as long as f_{xy} and f_{yx} are <u>CONTINUOUS</u>.

$\Delta x = 0.01, \quad \Delta y = -0.03$

6. (8 points) Let $z = \sqrt{7 - x^2 + y^3}$. Use differentials to estimate the change in z as (x, y) moves from $(2, 1)$ to $(2.01, 0.97)$.

$$\frac{\partial z}{\partial x} = \frac{1}{2}(7-x^2+y^3)^{-1/2}(-2x) = \frac{-x}{\sqrt{7-x^2+y^3}}, \quad \left.\frac{\partial z}{\partial x}\right|_{(2,1)} = \frac{-2}{2} = -1$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(7-x^2+y^3)^{-1/2}(3y^2) = \frac{3y^2}{2\sqrt{7-x^2+y^3}}, \quad \left.\frac{\partial z}{\partial y}\right|_{(2,1)} = \frac{3}{4}$$

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

$$\Rightarrow \Delta z \approx (-1)(0.01) + \left(\frac{3}{4}\right)(-0.03)$$

$$= \boxed{-0.0325}$$

7. (8 points) Find the linearization of $f(x, y, z) = \tan^{-1}(x^2 + 6y + 4z)$ at $(x,y,z) =$ $(1, 0, 0)$. Then use your linearization to approximate $f(0.9, 0.1, 0.1)$.

$$f_x(x,y,z) = \frac{1}{1+(x^2+6y+4z)^2} \cdot \frac{2x}{1}, \quad f_x(1,0,0) = \frac{2}{2} = 1$$

$$f_y(x,y,z) = \frac{1}{1+(x^2+6y+4z)^2} \cdot \frac{6}{1}, \quad f_y(1,0,0) = \frac{6}{2} = 3$$

$$f_z(x,y,z) = \frac{1}{1+(x^2+6y+4z)^2} \cdot \frac{4}{1}, \quad f_z(1,0,0) = \frac{4}{2} = 2$$

$$f(1,0,0) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\boxed{L(x,y,z) = \frac{\pi}{4} + (x-1) + 3y + 2z}$$

$$L(0.9, 0.1, 0.1)$$

$$= \frac{\pi}{4} - 0.1 + 0.3 + 0.2$$

$$= \boxed{\frac{\pi}{4} + 0.4}$$

8. (8 points) Suppose θ is implicitly defined as a function of x and y by the equation $y - x\tan\theta = 0$. Determine $\partial\theta/\partial x$ and $\partial\theta/\partial y$.

$$F(x,y,\theta) = y - x\tan\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{-F_x}{F_\theta} = \frac{-(-\tan\theta)}{-x\sec^2\theta} = \boxed{\frac{-\tan\theta}{x\sec^2\theta}}$$

$$\frac{\partial\theta}{\partial y} = \frac{-F_y}{F_\theta} = \frac{-1}{-x\sec^2\theta} = \boxed{\frac{1}{x\sec^2\theta}}$$

9. (8 points) Let $w = xy\cos z$, where $x = t$, $y = t^2$, and $z = \sin^{-1} t$. Use the appropriate chain rule to find a formula for dw/dt.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\boxed{\frac{dw}{dt} = (y\cos z)(1) + (x\cos z)(2t) - (xy\sin z)\left(\frac{1}{\sqrt{1-t^2}}\right)}$$

10. (2 points) At any point where f is differentiable, the directional derivative is greatest in the direction of the GRADIENT VECTOR.

11. (8 points) Let $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$. Determine a unit vector in the direction of the maximum increase from the point $(5,-5,5)$.

$$\vec{\nabla} f(x,y,z) = \left(\frac{1}{y} - \frac{z}{x^2}\right)\hat{i} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\hat{j} + \left(\frac{1}{x} - \frac{y}{z^2}\right)\hat{k}$$

$$\vec{\nabla} f(5,-5,5) = \left(-\frac{1}{5} - \frac{1}{5}\right)\hat{i} + \left(\frac{1}{5} - \frac{1}{5}\right)\hat{j} + \left(\frac{1}{5} + \frac{1}{5}\right)\hat{k}$$

$$= -\frac{2}{5}\hat{i} + \frac{2}{5}\hat{k}$$

UNIT VECTOR IN DIRECTION OF $-\hat{i}+\hat{k}$ $= \dfrac{\vec{\nabla} f(5,-5,5)}{\|\vec{\nabla} f(5,-5,5)\|} = \boxed{-\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{k}}$

12. (8 points) Let $g(x,y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$. Determine all points for which $\nabla g(x,y) = \vec{0}$.

$$\vec{\nabla} g(x,y) = (2x + 2y + 4)\hat{i} + (2x - 8y - 6)\hat{j}$$

$$\vec{\nabla} g(x,y) = \vec{0} \Rightarrow \begin{array}{r} 2x + 2y = -4 \\ 2x - 8y = 6 \\ \hline 10y = -10 \\ y = -1 \\ 2x - 2 = -4 \\ 2x = -2 \\ x = -1 \end{array} \Rightarrow \boxed{(x,y) = (-1,-1)}$$

13. (2 points) Suppose $f(x,y)$ is a differentiable on $\mathbb{R}^2$. $\nabla f(x_0,y_0)$ is normal to the LEVEL CURVE passing through (x_0,y_0).

14. (14 points) Consider the surface described by the equation $4x^2 - 2y^2 + z^2 = 12$.

(a) Identify the surface.

$4x^2 + z^2 = 12 + 2y^2 \Rightarrow$ ELLIPTICAL $\boxed{\text{HYPERBOLOID OF ONE-SHEET}}$

(b) Show that the point $(2, 2, 2)$ is on the surface.

$4(2)^2 - 2(2)^2 + (2)^2 = 16 - 8 + 4 = 12$ ✓

(c) Find a vector normal to the surface at the point $(2, 2, 2)$.

$F(x,y,z) = 4x^2 - 2y^2 + z^2$

$\vec{\nabla} F(2,2,2)$ IS NORMAL TO THE LEVEL SURFACE $F(x,y,z) = F(2,2,2)$.

$\vec{\nabla} F(x,y,z) = 8x\,\hat{i} - 4y\,\hat{j} + 2z\,\hat{k}$

$\vec{n} = \vec{\nabla} F(2,2,2) = 16\hat{i} - 8\hat{j} + 4\hat{k}$

(d) Find an equation of the plane tangent to the surface at $(2, 2, 2)$.

I'LL USE $\vec{n} = 4\hat{i} - 2\hat{j} + \hat{k}$

$4x - 2y + z = 4(2) - 2(2) + (2) = 8 - 4 + 2 = 6$

$4x - 2y + z = 6$

(e) Find a set of parametric equations for the line normal to the surface at $(2, 2, 2)$.

POINT $(2,2,2)$

DIRECTION: $4\hat{i} - 2\hat{j} + \hat{k}$

$x = 2 + 4t$
$y = 2 - 2t$
$z = 2 + t$