Math 233 - Final Exam A

May 9, 2023

Name key

Score ___________

Show all work to receive full credit. Supply explanations where necessary. This portion of the test is due May 11. You must work individually.

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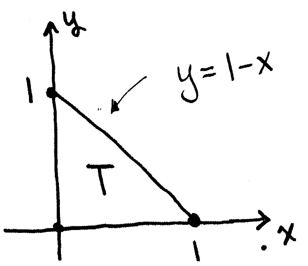
1. (10 points) Let C be the closed curve in the plane made up of three line segments: from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$. According to Green's theorem,

$$\int_C x^2y^3\,dx + x^2y\,dy = \iint_T (2xy - 3x^2y^2)\,dA,$$

where T is the triangular region inside the closed curve C. Evaluate the double integral.



$$\iint_T (2xy - 3x^2y^2)\,dA$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} (2xy - 3x^2y^2)\,dy\,dx$$

$$= \int_0^1 \left(xy^2 - x^2y^3\right)\Big|_0^{1-x}\,dx$$

$$= \int_0^1 \left[x(1-x)^2 - x^2(1-x)^3\right]dx = \int_0^1 \left(x - 2x^2 + x^3 - x^2 + 3x^3 - 3x^4 + x^5\right)dx$$

$$= \int_0^1 \left(x - 3x^2 + 4x^3 - 3x^4 + x^5\right)dx = \frac{1}{2} - 1 + 1 - \frac{3}{5} + \frac{1}{6}$$

$$= \boxed{\frac{1}{15}}$$

2. (10 points) Let $\vec{F}(x,y,z) = z^2\,\hat{\imath} + 2y\,\hat{\jmath} + 2xz\,\hat{k}$. Consider the line integral

$$\int_C \vec{F}(x,y,z)\cdot d\vec{r}.$$

(a) Evaluate the line integral when C is the line segment from $\underbrace{(0,0,0) \text{ to } (1,1,2)}$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C z^2dx + 2y\,dy + 2xz\,dz$$

$C:\ x = t,\quad y = t,\quad z = 2t \qquad 0 \le t \le 1.$

$$= \int_0^1 4t^2dt + 2t\,dt + 4t^2(2dt)$$

$dx = dy = dt$

$dz = 2dt$

$$= \int_0^1 (12t^2 + 2t)\,dt = 4 + 1 = \boxed{5}$$

(b) Evaluate the line integral when C is the graph of $\underbrace{\vec{r}(t) = t\,\hat{\imath} + t^2\hat{\jmath} + 2t^3\,\hat{k}}$ from $(0,0,0)$ to $(1,1,2)$.

$d\vec{r} = dt\,\hat{\imath} + 2t\,dt\,\hat{\jmath} + 6t^2dt\,\hat{k}$

$$\int_C z^2dx + 2y\,dy + 2xz\,dz$$

$$= \int_0^1 4t^6dt + 2t^2(2t\,dt) + (4t^4)(6t^2dt)$$

$$= \int_0^1 (28t^6 + 4t^3)\,dt = 4 + 1 = \boxed{5}$$

(c) Let $f(x,y,z) = xz^2 + y^2 + 3$. Show that $\nabla f(x,y,z) = \vec{F}(x,y,z)$.

$$\vec{\nabla} f(x,y,z) = f_x\,\hat{\imath} + f_y\,\hat{\jmath} + f_z\,\hat{k}$$

$$= z^2\,\hat{\imath} + 2y\,\hat{\jmath} + 2xz\,\hat{k} \quad ✓$$

(d) Using the function f from part (c), compute $f(1,1,2) - f(0,0,0)$.

$$f(1,1,2) - f(0,0,0) = [4+1+3] - [0+0+3]$$

$$= \boxed{5}$$