Math 233 - Final Exam B Name key

May 11, 2023 Score ______

Show all work to receive full credit. Supply explanations where necessary.

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1. (10 points) Two planes are described by the equations below.

$$P_1 : 3x - 4y + 5z = 8 \qquad P_2 : 7(x-4) + 3y - 3(z-9) = 0$$

Show that the planes are not parallel, and then find the measure of the angle between them. Write your final answer in degrees, rounded to the nearest hundredth.

$\vec{n}_1 = 3\hat{i} - 4\hat{j} + 5\hat{k}$

$\vec{n}_2 = 7\hat{i} + 3\hat{j} - 3\hat{k}$

$\vec{n}_1$ & $\vec{n}_2$ ARE NOT PARALLEL ($\vec{n}_1 \neq k\vec{n}_2$ FOR ANY k).

NORMALS NOT PARALLEL $\Rightarrow$ PLANES NOT PARALLEL.

$\vec{n}_1 \cdot \vec{n}_2 = 21 - 12 - 15 = -6$

$\|\vec{n}_1\| = \sqrt{9+16+25} = \sqrt{50}$

$\|\vec{n}_2\| = \sqrt{49+9+9} = \sqrt{67}$

ANGLE BETWEEN P_1 & P_2

$= \cos^{-1}\left(\dfrac{6}{\sqrt{50}\sqrt{67}}\right) \approx 84.05^\circ$

2. (10 points) Find the arc-length parameterization (starting from $t = 0$) for the curve described by the vector-valued function

$$\vec{r}(t) = 5\cos t\,\hat{\imath} + 12t\,\hat{\jmath} + 5\sin t\,\hat{k}.$$

$\vec{r}'(t) = -5\sin t\,\hat{i} + 12\hat{j} + 5\cos t\,\hat{k}$

$\|\vec{r}'(t)\| = \sqrt{25\sin^2 t + 144 + 25\cos^2 t}$

$= \sqrt{169} = 13$

$s(t) = \displaystyle\int_0^t 13\,d\tau = 13\tau \Big|_0^t = 13t$

$s = 13t$

$\Downarrow$

$t = s/13$

$\vec{R}(s) = 5\cos\frac{s}{13}\,\hat{i} + \frac{12}{13}s\,\hat{j} + 5\sin\frac{s}{13}\,\hat{k}$

3. (10 points) A small projectile is launched from ground level with an initial speed of 98 m/s. Find the possible launch angles so that the range of the projectile is 490 m. (You may need to use a double angle trig formula.)

$\vec{r}(t) = 98\cos\theta\, t\ \hat{i} + (-4.9t^2 + 98\cos\theta\, t)\ \hat{j}$

$-4.9t^2 + 98\sin\theta\, t = 0$ WHEN $98\cos\theta\, t = 490$

$t(-4.9t + 98\sin\theta) = 0$ $\qquad t = \dfrac{490}{98\cos\theta}$

$\cancel{t=0}$ OR $t = \dfrac{98\sin\theta}{4.9}$

$\dfrac{490}{98\cos\theta} = \dfrac{98\sin\theta}{4.9} \Rightarrow \sin\theta\cos\theta = \dfrac{1}{4}$

$2\sin\theta\cos\theta = \dfrac{1}{2}$

$\sin 2\theta = \dfrac{1}{2}$

$2\theta = \dfrac{\pi}{6}$ OR $\dfrac{5\pi}{6}$

$\theta = \dfrac{\pi}{12}$ OR $\dfrac{5\pi}{12}$

4. (10 points) Consider the surface described by the equation

$$x\ln(z^2) + 5y^2e^{2x} = 42 + yz\cos(4x).$$

(a) Find an equation of the plane tangent to the surface at the point (0, 3, 1).

$F(x,y,z) = x\ln(z^2) + 5y^2e^{2x} - yz\cos(4x)$

$\vec{\nabla} F(x,y,z) = \left[\ln(z^2) + 10y^2e^{2x} + 4yz\sin(4x)\right]\hat{i} + \left[10y\,e^{2x} - z\cos(4x)\right]\hat{j} + \left[\dfrac{2x}{z} - y\cos(4x)\right]\hat{k}$

$\vec{n} = \vec{\nabla}F(0,3,1) = 90\hat{i} + 29\hat{j} - 3\hat{k}$

TANGENT PLANE: $90x + 29(y-3) - 3(z-1) = 0$

(b) Find a set of parametric equations for the line normal to the surface at the point (0, 3, 1).

$x = 90t$

$y = 29t + 3$

$z = -3t + 1$

5. (10 points) Find the critical points of $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$. Then use the second partials test to classify the critical points and find the extreme values.

$f_x(x,y) = 3x^2 - 6x - 9 = 0 \Rightarrow 3(x-3)(x+1) = 0 \Rightarrow x = 3,\ x = -1$

$f_y(x,y) = 3y^2 - 6y = 0 \Rightarrow 3y(y-2) = 0 \Rightarrow y = 0,\ y = 2$

CRIT PTS ARE $(3,0)$, $(3,2)$, $(-1,0)$, $(-1,2)$

$$D = \begin{vmatrix} 6x-6 & 0 \\ 0 & 6y-6 \end{vmatrix} = (6x-6)(6y-6)$$

$(3,0)$: $D(3,0) < 0 \Rightarrow$ $\boxed{(3,0,-27) \text{ IS A SADDLE POINT.}}$

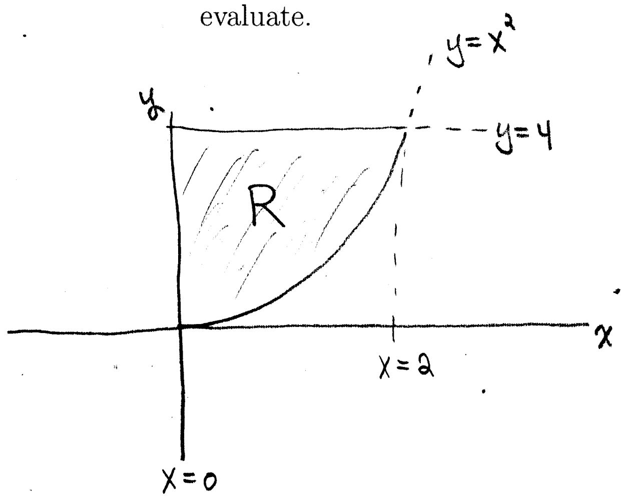
$(3,2)$: $D(3,2) > 0$ AND $f_{xx}(3,2) > 0 \Rightarrow$ $f(3,2) = -31$ IS A REL. MINIMUM

$(-1,0)$: $D(-1,0) > 0$ AND $f_{xx}(-1,0) < 0 \Rightarrow$ $f(-1,0) = 5$ IS A REL. MAXIMUM

$(-1,2)$: $D(-1,2) < 0 \Rightarrow$ $\boxed{(-1,2,1) \text{ IS A SADDLE POINT.}}$

6. (10 points) Sketch the region of integration, reverse the order of integration, and evaluate.

$$\int_0^2 \int_{x^2}^4 x e^{y^2}\, dy\, dx$$



$$\int_{y=0}^{y=4} \int_{x=0}^{x=\sqrt{y}} x e^{y^2}\, dx\, dy$$

$$= \int_0^4 \frac{1}{2} (\sqrt{y})^2 e^{y^2}\, dy = \frac{1}{2} \int_0^4 y e^{y^2}\, dy$$

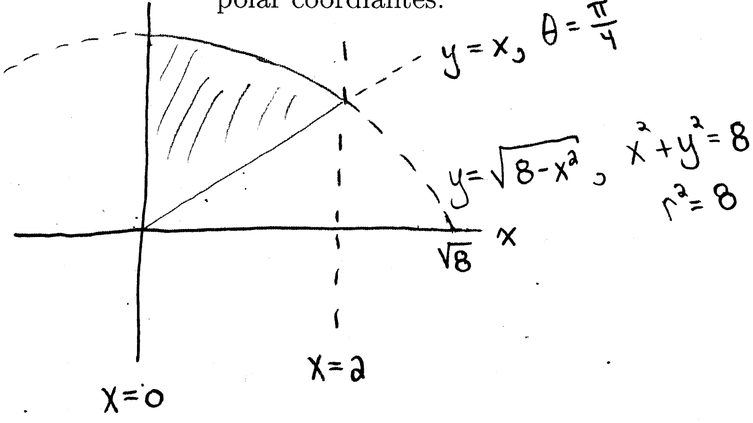
$u = y^2, \quad \frac{1}{2} du = y\, dy$

$$\frac{1}{4} \int_0^{16} e^u\, du = \boxed{\frac{1}{4}\left(e^{16} - 1\right)}$$

7. (10 points) The volume of a solid is given by

$$\int_0^2 \int_x^{\sqrt{8-x^2}} \frac{1}{5 + x^2 + y^2}\, dy\, dx.$$

Sketch the region of integration, and then evaluate the integral by first converting to polar coordiantes.



$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\sqrt{8}} \frac{r}{5+r^2}\, dr\, d\theta$$

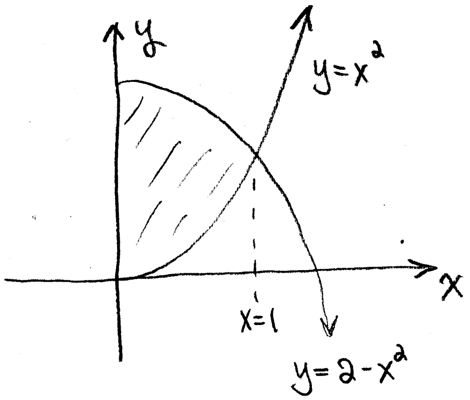
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{\sqrt{8}} \frac{r}{5+r^2}\, dr$$

$u = 5 + r^2$

$$= \frac{\pi}{4} \left(\frac{1}{2} \ln (5 + r^2) \Big|_0^{\sqrt{8}} \right)$$

$$= \boxed{\frac{\pi}{8} \left[\ln 13 - \ln 5 \right] \approx 0.375}$$

8. (10 points) A region in space lies in the first octant (where $x, y, z \geq 0$) inside the cylinders $y = x^2$ and $y = 2 - x^2$, above the plane $z = 0$, and below the plane $z = 2 + y$. The volume of the region is $4\,\text{units}^3$. Use a triple integral to find the average value of $f(x, y, z) = x^2$ over the region.



$$2-x^2 = x^2 \Rightarrow x = 1$$

Average value $=$

$$\frac{1}{4}\int_{x=0}^{x=1}\int_{y=x^2}^{y=2-x^2}\int_{z=0}^{z=2+y} x^2 \, dz\, dy\, dx$$

$$= \frac{1}{4}\int_0^1\int_{x^2}^{2-x^2} (2+y)x^2 \, dy\, dx$$

$$= \frac{1}{4}\int_0^1 \left(2y + \frac{y^2}{2}\right)x^2 \Big|_{x^2}^{2-x^2} dx$$

$$= \frac{1}{4}\int_0^1 \left[\left(2(2-x^2) + \frac{(2-x^2)^2}{2}\right)x^2 - \left(2x^2 + \frac{x^4}{2}\right)x^2\right] dx$$

$$= \frac{1}{4}\int_0^1 \left(4x^2 - 2x^4 + 2x^2 - 2x^4 + \frac{x^6}{2} - 2x^4 - \frac{x^6}{2}\right) dx$$

$$= \frac{1}{4}\int_0^1 (6x^2 - 6x^4)\, dx = \frac{1}{4}\left[\frac{6}{3} - \frac{6}{5}\right] = \boxed{\frac{1}{5}}$$