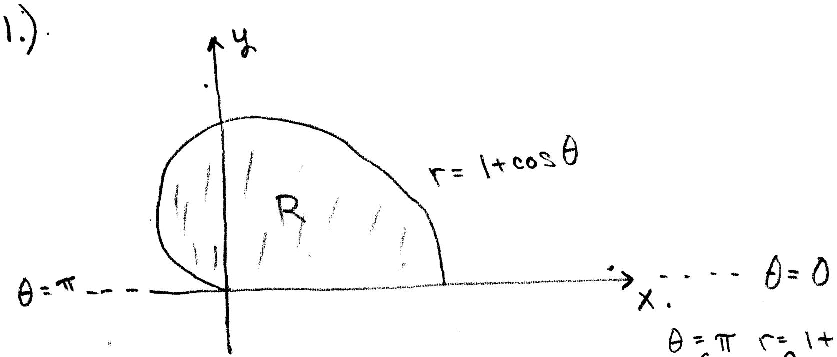
MTH 233 - Assignment #11 Key

1.)



$$\text{Area} = \iint_R 1\,dA = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{(1+\cos\theta)^2}{2}\,d\theta = \int_0^{\pi}\left(\frac{1}{2}+\cos\theta+\frac{1}{2}\cos^2\theta\right)d\theta$$

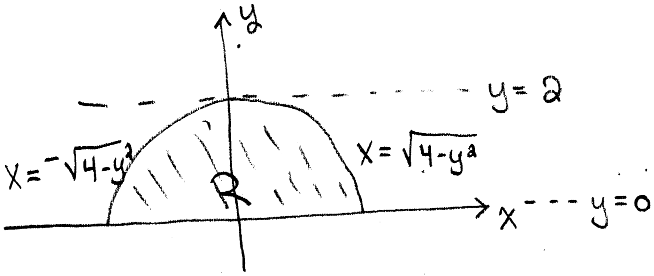
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_0^{\pi}\left(\frac{3}{4}+\cos\theta+\frac{1}{4}\cos 2\theta\right)d\theta$$

$$= \frac{3}{4}\theta + \sin\theta + \frac{1}{8}\sin 2\theta\,\Big|_0^{\pi} = \boxed{\frac{3\pi}{4}}$$

2.) $$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2+y^2)^2 \, dx \, dy$$

$$x = \pm\sqrt{4-y^2} \Rightarrow x^2 = 4-y^2 \Rightarrow x^2+y^2 = 4$$



CONVERT TO POLAR ...

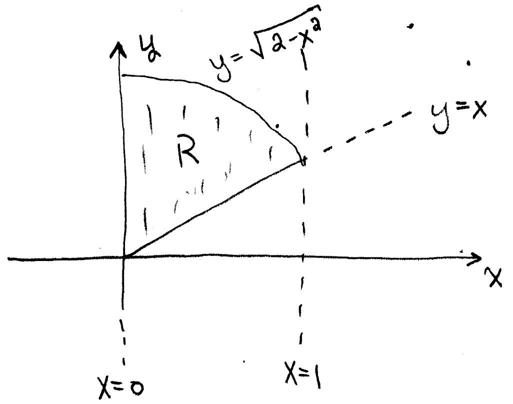
$$\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2} r^4 \, r \, dr \, d\theta = \pi \int_0^2 r^5 \, dr$$

$$= \frac{\pi}{6} r^6 \Big|_0^2 = \frac{64\pi}{6}$$

$$= \boxed{\frac{32\pi}{3}}$$

3.) $$\int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y)\,dy\,dx$$

$$y=\sqrt{2-x^2} \Rightarrow y^2 = 2-x^2 \Rightarrow x^2+y^2=2$$



Convert to polar...

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\sqrt{2}} (r\cos\theta + 2r\sin\theta)\, r\,dr\,d\theta$$

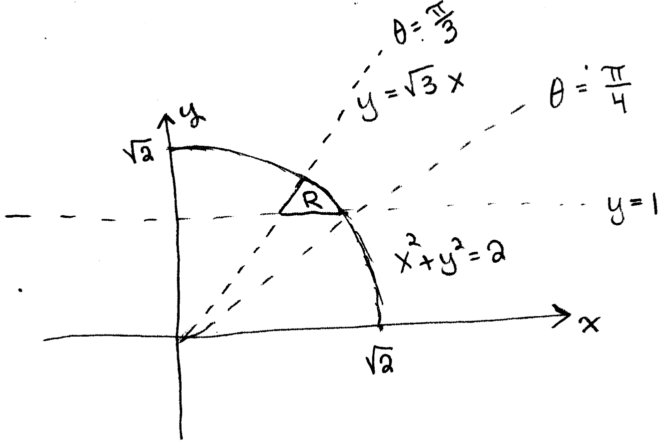
$$= \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} r^2(\cos\theta + 2\sin\theta)\,dr\,d\theta = \frac{2\sqrt{2}}{3}\int_{\pi/4}^{\pi/2} (\cos\theta + 2\sin\theta)\,d\theta$$

$$= \frac{2\sqrt{2}}{3}(\sin\theta - 2\cos\theta)\Big|_{\pi/4}^{\pi/2} = \frac{2\sqrt{2}}{3}\left[(1-0) - \left(\frac{\sqrt{2}}{2} - \sqrt{2}\right)\right]$$

$$= \frac{2\sqrt{2}}{3}\left(1+\frac{\sqrt{2}}{2}\right)$$

$$= \boxed{\frac{2\sqrt{2}}{3} + \frac{2}{3}}$$

4.)



$$\frac{y}{x} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

$$\frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$y = 1 \Rightarrow r\sin\theta = 1$$
$$\Rightarrow r = \csc\theta$$

$$\text{Area} = \iint_R 1\, dA = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{3}} \int_{r=\csc\theta}^{r=\sqrt{2}} r \cdot dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{r^2}{2}\Bigg|_{\csc\theta}^{\sqrt{2}} d\theta = \int_{\pi/4}^{\pi/3} \left(1 - \frac{1}{2}\csc^2\theta\right) d\theta$$

$$= \theta + \frac{1}{2}\cot\theta \Bigg|_{\pi/4}^{\pi/3}$$

$$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{1}{2}\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{4}\right)$$

$$= \boxed{\frac{\pi}{12} + \frac{1}{2}\left(\frac{1}{\sqrt{3}} - 1\right) \approx 0.0504745}$$

5.) See next page for graph.

See next page for graph.

$$\text{Area} = \iint_R 1\,dA = \int_0^{\pi/6}\int_0^{3\sin\theta} r\,dr\,d\theta + \int_{\pi/6}^{\pi/2}\int_0^{\sqrt{3}\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/6} \frac{9\sin^2\theta}{2}\,d\theta + \int_{\pi/6}^{\pi/2} \frac{3\cos^2\theta}{2}\,d\theta$$

$$= \frac{9}{4}\int_0^{\pi/6} (1-\cos 2\theta)\,d\theta + \frac{3}{4}\int_{\pi/6}^{\pi/2} (1+\cos 2\theta)\,d\theta$$

$$= \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6} + \frac{3}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

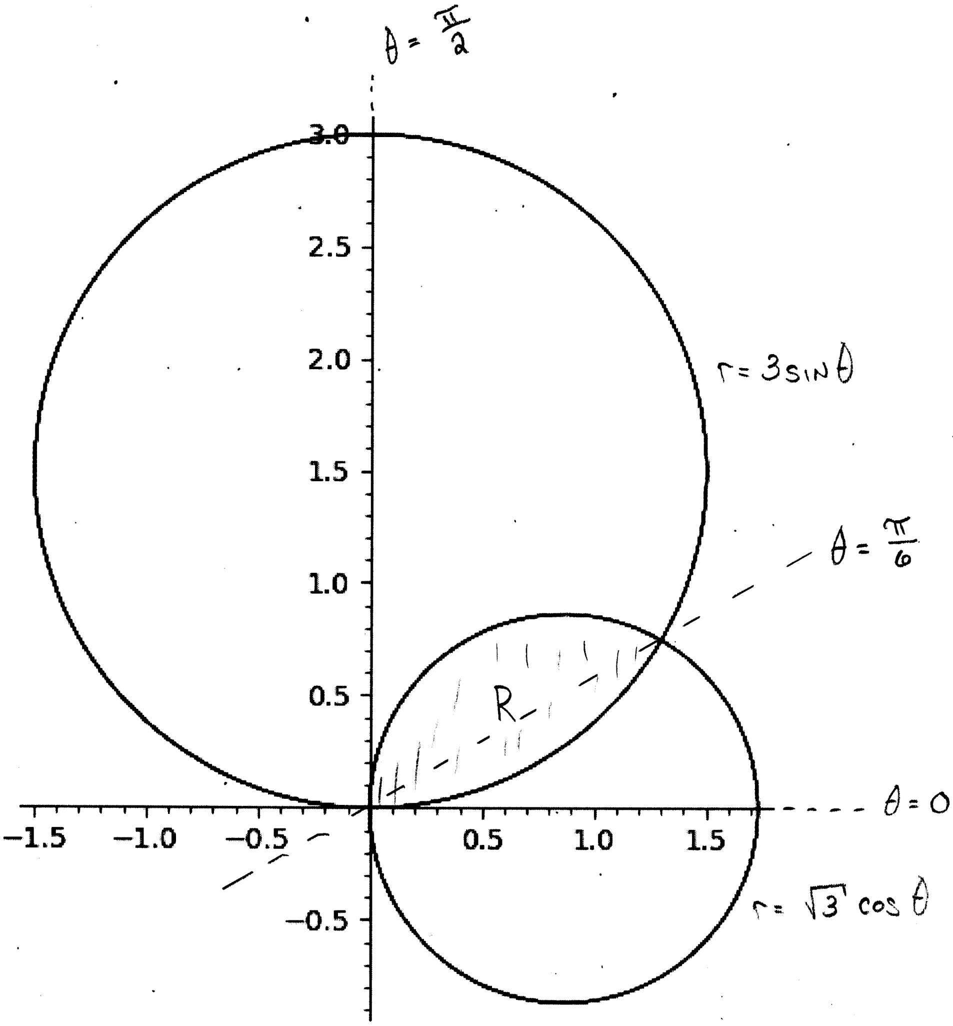
$$= \frac{9}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] + \frac{3}{4}\left[\frac{\pi}{2} - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)\right]$$

$$= \frac{3\pi}{8} - \frac{9\sqrt{3}}{16} + \frac{3\pi}{8} - \frac{\pi}{8} - \frac{3\sqrt{3}}{16}$$

$$= \boxed{\frac{5\pi}{8} - \frac{3\sqrt{3}}{4} \approx 0.664457}$$

The other order...

$$\int_{r=0}^{r=3/2}\int_{\theta=\sin^{-1}(r/3)}^{\theta=\cos^{-1}(r/\sqrt{3})} r\,d\theta\,dr = \cdots = \frac{5\pi}{8} - \frac{3\sqrt{3}}{4}$$

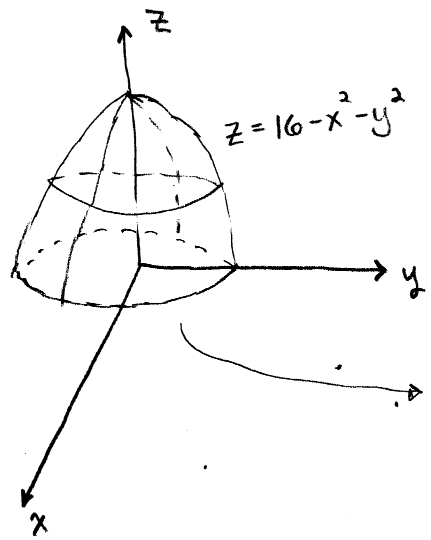


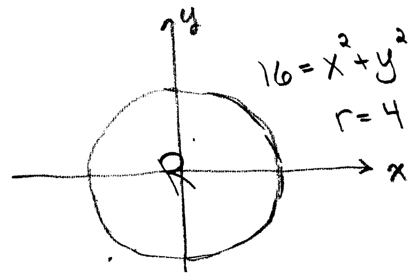
$3\sin\theta = \sqrt{3}\cos\theta$

$\Rightarrow \tan\theta = \frac{\sqrt{3}}{3}$

$\Rightarrow \theta = \frac{\pi}{6}$

6.)





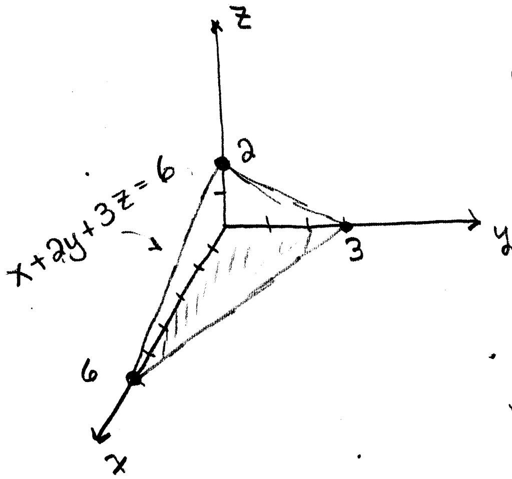
$$\text{AVERAGE VALUE} = \frac{1}{\text{VOLUME OF } S} \iiint_S \left(1 + x^2 + y^2 + z^3\right)\, dV$$

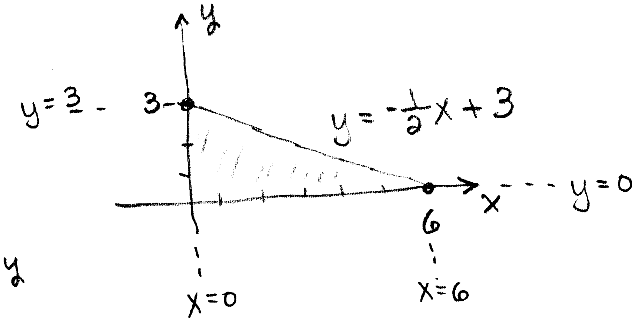
$$\text{VOLUME OF } S = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} \int_{z=0}^{z=16-r^2} r\, dz\, dr\, d\theta = 128\pi$$

$$\iiint_S \left(1 + x^2 + y^2 + z^3\right) dV = \int_0^{2\pi} \int_0^{4} \int_0^{16-r^2} r\left(1 + r^2 + z^3\right) dz\, dr\, d\theta = \frac{798592\pi}{15}$$

$$\text{AVERAGE VALUE} = \frac{\frac{798592}{15}\pi}{128\pi} = \frac{6239}{15}$$

7.)





$$\text{VOLUME OF } T = \iiint_T 1\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=-\frac{1}{2}x+3} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} 1\, dz\, dy\, dx = 6$$

$$\iiint_T (x+y+z)\, dV = \int_0^6 \int_0^{-\frac{1}{2}x+3} \int_0^{\frac{1}{3}(6-x-2y)} (x+y+z)\, dz\, dy\, dx$$

$$= \frac{33}{2}$$

AVERAGE VALUE OF $f(x,y,z) = x+y+z$ OVER T

$$= \frac{33/2}{6} = \frac{33}{12} = \boxed{\frac{11}{4}}$$