Math 233 - Assignment 1

January 18, 2024

Name KEY ______________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due January 25.

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1. The vector $\vec{v}$ has initial point $(-2, 5)$ and terminal point $(3, -1)$. Find a unit vector in the direction of $\vec{v}$.

   Solution

   In component form, $\vec{v} = \langle 3 - (-2), -1 - 5 \rangle = \langle 5, -6 \rangle = 5\hat{\imath} - 6\hat{\jmath}$.

   $\|\vec{v}\| = \sqrt{5^2 + (-6)^2} = \sqrt{61}$

   $$\frac{\vec{v}}{\|\vec{v}\|} = \frac{5}{\sqrt{61}}\,\hat{\imath} - \frac{6}{\sqrt{61}}\,\hat{\jmath}$$

2. Find a vector of magnitude 7 whose direction is opposite that of $\langle 3, -4 \rangle$.

   Solution

   Let $\vec{v} = \langle 3, -4 \rangle = 3\hat{\imath} - 4\hat{\jmath}$.

   $\|\vec{v}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

   $$-\frac{7\vec{v}}{\|\vec{v}\|} = -\frac{21}{5}\,\hat{\imath} + \frac{28}{5}\,\hat{\jmath}$$

3. The vector $\vec{w}$ has initial point $P\,(1, 1)$ and terminal point Q. Q lies on the x-axis and left of the initial point. Find the coordinates of Q if $\|\vec{w}\| = \sqrt{10}$.

   Solution

   The point Q has the form $(x, 0)$ for some real number $x < 1$. Therefore, $\vec{w} = (x-1)\,\hat{\imath} - \hat{\jmath}$.

   $\|\vec{w}\| = \sqrt{10} = \sqrt{(x-1)^2 + (-1)^2}$, and it follows that $(x-1)^2 + 1 = 10$.

   $(x-1)^2 = 9 \implies x = 4$ or $x = -2$

   Since $x < 1$, we must have $x = -2$ and $Q\,(-2, 0)$.

4. Suppose $\vec{u}$ and $\vec{v}$ are nonzero, unequal vectors. Also suppose that $\vec{a} = 2\vec{u} - 4\vec{v}$ and $\vec{b} = 3\vec{u} - 7\vec{v}$. Find scalars α and β so that $\alpha\vec{a} + \beta\vec{b} = \vec{u} - \vec{v}$.

   Solution

   $\alpha\vec{a} + \beta\vec{b} = \alpha(2\vec{u} - 4\vec{v}) + \beta(3\vec{u} - 7\vec{v}) = (2\alpha + 3\beta)\vec{u} - (4\alpha + 7\beta)\vec{v}$.

   To satisy the required condition, we must have $2\alpha + 3\beta = 1$ and $4\alpha + 7\beta = 1$.

   Solve the system of equations to get $\alpha = 2$ and $\beta = -1$.

5. Let $\vec{a}$ be the standard-position vector with terminal point at $(2, 5)$. Let $\vec{b}$ be the vector with initial point at $(-1, 3)$ and terminal point $(1, 0)$. Compute $\|\vec{a} - 3\vec{b} + 14\hat{\imath} - 14\hat{\jmath}\|$.

   Solution

   $\vec{a} = 2\hat{\imath} + 5\hat{\jmath}$ and $\vec{b} = 2\hat{\imath} - 3\hat{\jmath}$.

   $\vec{a} - 3\vec{b} + 14\hat{\imath} - 14\hat{\jmath} = 2\hat{\imath} + 5\hat{\jmath} - 6\hat{\imath} + 9\hat{\jmath} + 14\hat{\imath} - 14\hat{\jmath} = 10\hat{\imath}$

   $\|\vec{a} - 3\vec{b} + 14\hat{\imath} - 14\hat{\jmath}\| = \|10\hat{\imath}\| = 10$.

6. Determine the vector $\vec{PM}$, where M is the midpoint of $P\,(5, 2, -9)$ and $Q\,(-7, 11, 3)$.

   Solution

   $\vec{PM} = \frac{1}{2}\vec{PQ} = \frac{1}{2}\langle -12, 9, 12\rangle = -6\hat{\imath} + \frac{9}{2}\hat{\jmath} + 6\hat{k}$

   Alternative approach: The midpoint is $M\,(-1, 13/2, -3)$. Now find the component form of $\vec{PM}$.

7. Let $P\,(x, y, z)$ be a point situated an at equal distance from the origin and from the point $(4, 1, 2)$. Show that the coordinates of P satisfy $8x + 2y + 4z = 21$.

   Solution

   Distance from P to origin $= \sqrt{x^2 + y^2 + z^2}$

   Distance from P to $(4, 1, 2) = \sqrt{(x-4)^2 + (y-1)^2 + (z-2)^2}$

   Distances are equal. Therefore, $x^2 + y^2 + z^2 = (x-4)^2 + (y-1)^2 + (z-2)^2$.

   Expand and simplify to get $8x + 2y + 4z = 21$.

8. Show that the points $P\,(1,0,1)$, $Q\,(0,1,1)$, and $R\,(1,1,1)$ are NOT collinear.

   Solution

   $\vec{PQ} = -\hat{\imath} + \hat{\jmath}$ and $\vec{QR} = \hat{\imath}$.

   The vectors $\vec{PQ}$ and $\vec{QR}$ are NOT parallel. Therefore P, Q, and R cannot be collinear.

9. Determine the vector of magnitude 13 that is parallel to $\vec{v} = 8\hat{\imath} - 7\hat{\jmath} + 12\hat{k}$.

   Solution

   $\|\vec{v}\| = \sqrt{8^2 + (-7)^2 + 12^2} = \sqrt{257}$

   $$\frac{13\vec{v}}{\sqrt{257}} = \frac{104}{\sqrt{257}}\hat{\imath} - \frac{91}{\sqrt{257}}\hat{\jmath} + \frac{156}{\sqrt{257}}\hat{k}$$

10. The vector $\vec{v}$ has magnitude 4 and the direction from $(4,5,-2)$ to $(3,8,-9)$. The vector $\vec{w}$ lies in the xy-plane, has length $\sqrt{8}$, and makes a $45°$ angle with the positive x-axis. Compute $\vec{v} - \vec{w}$.

    Solution

    The vector from $P\,(4,5,-2)$ to $Q\,(3,8,-9)$ is $\vec{PQ} = -\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$. It has magnitude $\|\vec{PQ}\| = \sqrt{(-1)^2 + 3^2 + (-7)^2} = \sqrt{59}$. Therefore $\vec{v} = \frac{4}{\sqrt{59}}(-\hat{\imath} + 3\hat{\jmath} - 7\hat{k})$.

    $\vec{w} = \sqrt{8}\cos(45°)\,\hat{\imath} + \sqrt{8}\sin(45°)\,\hat{\jmath} = 2\hat{\imath} + 2\hat{\jmath}$

    $\vec{v} - \vec{w} = (-\frac{4}{\sqrt{59}} - 2)\,\hat{\imath} + (\frac{12}{\sqrt{59}} - 2)\,\hat{\jmath} - \frac{28}{\sqrt{59}}\,\hat{k} \approx -2.52076\,\hat{\imath} - 0.43773\,\hat{\jmath} - 3.64529\,\hat{k}$