Math 233 - Assignment 2

Name KEY ____________

January 25, 2024

Show all work to receive full credit. Supply explanations when necessary. This assignment is due February 1.

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1. Let $\vec{v} = 7\hat{\imath} - 8\hat{\jmath} + 3\hat{k}$ and $\vec{w} = -5\hat{\imath} + 6\hat{k}$. Find the measure of the angle between $\vec{v}$ and $\vec{w}$. Write your final answer in degrees, rounded to the nearest thousandth.

   Solution

   The angle θ satifies $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \dfrac{-35+18}{\sqrt{49+64+9}\sqrt{25+36}} = -\dfrac{17}{\sqrt{7442}}$. It follows that $\theta \approx 101.365°$.

2. Let $\vec{v} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{u} = 4\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$. Now let $\vec{w} = \text{proj}_{\vec{v}}\,\vec{u}$ and $\vec{x} = \vec{u} - \vec{w}$. Compute $\vec{w}$ and $\vec{x}$ and show that they are orthogonal.

   Solution

   $\vec{w} = \text{proj}_{\vec{v}}\,\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \dfrac{18}{9}\,\vec{v} = 2\vec{v} = 4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$. It follows that $\vec{x} = \vec{u} - \vec{w} = 4\hat{\jmath} + 2\hat{k}$.

   Finally, we show that $\vec{w}$ and $\vec{x}$ are orthogonal: $\vec{w}\cdot\vec{x} = 0 - 8 + 8 = 0.$ ✓.

3. If $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w}$, must it be true that $\vec{v} = \vec{w}$?

   Solution

   No way! If $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w}$, then $\vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{w} = 0$. This simply implies that $\vec{u}\cdot(\vec{v} - \vec{w}) = 0$. Therefore, $\vec{v}$ and $\vec{w}$ can be any vectors as long as their difference is orthogonal to $\vec{u}$.

   For example, let $\vec{u} = 5\hat{\imath} + 3\hat{\jmath} + 8\hat{k}$, $\vec{v} = 8\hat{\jmath}$, and $\vec{w} = 3\hat{k}$.

4. Suppose $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Prove that $\vec{u}$ is orthogonal to $5\vec{v} - 3\vec{w}$.

   Solution

   We know that $\vec{u}\cdot\vec{v} = 0$ and $\vec{u}\cdot\vec{w} = 0$. It follows that $\vec{u}\cdot(5\vec{v} - 3\vec{w}) = 5\vec{u}\cdot\vec{v} - 3\vec{u}\cdot\vec{w} = 5(0) - 3(0) = 0.$ ✓

5. Find the measure of the angle that $\vec{u} = -8\hat{\imath} + 7\hat{\jmath} + 2\hat{k}$ makes with the positive y-axis. Write your final answer in degrees, rounded to the nearest thousandth.

   Solution

The angle β satifies $\cos\beta = \dfrac{\vec{u}\cdot\hat{\jmath}}{\|\vec{u}\|\|\hat{\jmath}\|} = \dfrac{7}{\sqrt{64+49+4}\,\sqrt{1}} = \dfrac{7}{\sqrt{117}}$. It follows that $\beta \approx 49.673°$.

6. Find a unit vector that is orthogonal to both $\vec{v} = 4\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\vec{u} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

   Solution

   $\vec{v}\times\vec{u} = 7\hat{\imath} - 13\hat{\jmath} - 11\hat{k}$, and $\|\vec{v}\times\vec{u}\| = \sqrt{339}$.

   $$\frac{\vec{v}\times\vec{u}}{\|\vec{v}\times\vec{u}\|} = \frac{7}{\sqrt{339}}\hat{\imath} - \frac{13}{\sqrt{339}}\hat{\jmath} - \frac{11}{\sqrt{339}}\hat{k}.$$

7. Find the area of the $\triangle ABC$, where $A\,(1,2,3)$, $B\,(0,-9,-4)$, and $C\,(-5,8,-3)$.

   Solution

   Area $= \frac{1}{2}\|\vec{AB}\times\vec{AC}\|$

   $\vec{AB} = -\hat{\imath} - 11\hat{\jmath} - 7\hat{k}$ and $\vec{AC} = -6\hat{\imath} + 6\hat{\jmath} - 6\hat{k}$.

   $\vec{AB}\times\vec{AC} = 108\hat{\imath} + 36\hat{\jmath} - 72\hat{k}$ and $\|\vec{AB}\times\vec{AC}\| = \sqrt{18144}$. It follows that the area is $\frac{1}{2}\sqrt{18144} \approx 67.35$ units^2.

8. Find parametric and symmetric equations for the line in space that passes through the points $P\,(8,9,-4)$ and $Q\,(6,-2,-4)$.

   Solution

   Let's use point $P\,(8,9,-4)$ and the direction vector $\vec{v} = \vec{PQ} = -2\hat{\imath} - 11\hat{\jmath} + 0\hat{k}$.

   Parametric equations: $x = -2t + 8, \quad y = -11t + 9, \quad z = -4$

   Symmetric equations: $\dfrac{x-8}{-2} = \dfrac{y-9}{-11}, \quad z = -4$

9. A line is described by the equations $\dfrac{x+5}{3} = 2y - 4 = -\dfrac{z}{6}$.
   Find a point on the line and a vector that is parallel to the line. Then write parametric equations for the line.

   Solution

   Rewrite the symmetric equations: $\dfrac{x+5}{3} = \dfrac{y-2}{1/2} = \dfrac{z-0}{-6}$.

   Now we can read the point $(-5,2,0)$ and direction vector $\vec{v} = 3\hat{\imath} + \frac{1}{2}\hat{\jmath} - 6\hat{k}$.

   Then parametric equations could be $x = 3t - 5, \quad y = \frac{1}{2}t + 2, \quad z = -6t$.