Assignment #4

1) $\dfrac{d\vec{r}}{dt} = \cos t\ \hat{i} + 5\sin t\ \hat{j} + e^{-t}\ \hat{k}$, $\vec{r}(0) = 2\hat{i} + 7\hat{j} + 4\hat{k}$

$$\vec{r}(t) = (\sin t + c_1)\hat{i} + (-5\cos t + c_2)\hat{j} + (-e^{-t} + c_3)\hat{k}$$

$$\vec{r}(0) = c_1\hat{i} + (-5 + c_2)\hat{j} + (-1 + c_3)\hat{k} = 2\hat{i} + 7\hat{j} + 4\hat{k}.$$

$$\Downarrow$$

$$c_1 = 2,\ c_2 = 12,\ c_3 = 5$$

$$\boxed{\vec{r}(t) = (2 + \sin t)\hat{i} + (12 - 5\cos t)\hat{j} + (5 - e^{-t})\hat{k}}$$

2) $\vec{r}'(t) = (-\sin t + \sin t + t\cos t)\hat{i}$

$$+ (\cos t - \cos t + t\sin t)\hat{j}$$

$$= t\cos t\ \hat{i} + t\sin t\ \hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{\underbrace{t^2\cos^2 t + t^2\sin^2 t}_{t^2}} = |t| = t,\ t \geq 0$$

$$\boxed{\hat{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \cos t\ \hat{i} + \sin t\ \hat{j}}$$

3) $\vec{r}(t) = \cos 5t\ \hat{i} - t\hat{j} - \sin 5t\ \hat{k}$

$\vec{r}\,'(t) = -5\sin 5t\ \hat{i} - \hat{j} - 5\cos 5t\ \hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{25\sin^2 5t + 1 + 25\cos^2 5t} = \sqrt{26}$

$$s = \int_0^t \sqrt{26}\, d\tau = \sqrt{26}\, t \Rightarrow t = \frac{s}{\sqrt{26}}$$

$$\boxed{\vec{R}(s) = \cos\left(\frac{5s}{\sqrt{26}}\right)\hat{i} - \frac{s}{\sqrt{26}}\hat{j} - \sin\left(\frac{5s}{\sqrt{26}}\right)\hat{k}}$$

$$\vec{R}\,'(s) = -\frac{5}{\sqrt{26}}\sin\left(\frac{5s}{\sqrt{26}}\right)\hat{i} - \frac{1}{\sqrt{26}}\hat{j} - \frac{5}{\sqrt{26}}\cos\left(\frac{5s}{\sqrt{26}}\right)\hat{k}$$

$$\|\vec{R}\,'(s)\| = \sqrt{\frac{25}{26}\sin^2\left(\frac{5s}{\sqrt{26}}\right)\hat{i} + \frac{1}{26} + \frac{25}{26}\cos^2\left(\frac{5s}{\sqrt{26}}\right)}$$

$$= \sqrt{\frac{26}{26}} = 1 \quad \checkmark$$

4) $\vec{r}\,'(t) = 6t\,\hat{i} + 8t\,\hat{j} + 12t^2\,\hat{k}$

$$\|\vec{r}\,'(t)\| = \sqrt{36t^2 + 64t^2 + 144t^4}$$
$$= \sqrt{100t^2 + 144t^4} = t\sqrt{100 + 144t^2}, \quad t \geq 0$$

$$S = \int_0^2 t\sqrt{100 + 144t^2}\,dt$$

$$u = 100 + 144t^2$$
$$du = 288t\,dt \Rightarrow \frac{1}{288}\,du = t\,dt$$

$$\frac{1}{288}\int_{100}^{676} \sqrt{u}\,du = \frac{1}{288}\left(\frac{2}{3}\right)u^{3/2}\Big|_{100}^{676}$$

$$= \frac{1}{432}\left[676^{3/2} - 100^{3/2}\right] = \frac{1}{432}\left(26^3 - 10^3\right)$$

$$= \frac{16576}{432} \approx 38.37$$

5) $\vec{r}(t) = (t^2 - t)\hat{i} + \frac{1}{6}(4t-1)^{3/2}\hat{j} + 5\hat{k}$

$\vec{r}\,'(t) = (2t-1)\hat{i} + \frac{1}{4}(4t-1)^{1/2}(4)\hat{j}$

$= (2t-1)\hat{i} + (4t-1)^{1/2}\hat{j}$

$\|\vec{r}\,'(t)\| = \sqrt{(2t-1)^2 + (4t-1)} = \sqrt{4t^2} = 2t, \quad t \geq 0$

$S = \int_1^t 2\tau \, d\tau = \tau^2 \Big|_1^t = t^2 - 1$

$\Rightarrow \quad t = \sqrt{s+1}, \quad s \geq 0$ (SINCE t STARTS AT 1)

$$\boxed{\vec{R}(s) = \left(s+1-\sqrt{s+1}\right)\hat{i} + \frac{1}{6}\left(4\sqrt{s+1}-1\right)^{3/2}\hat{j} + 5\hat{k}}$$

6) $\vec{r}(t) = -\cos 3t\,\hat{i} - \sin 3t\,\hat{j} + 4t\,\hat{k}$

$\vec{r}\,'(t) = 3\sin 3t\,\hat{i} - 3\cos 3t\,\hat{j} + 4\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 16} = 5$

$\hat{T}(t) = \frac{3}{5}\sin 3t\,\hat{i} - \frac{3}{5}\cos 3t\,\hat{j} + \frac{4}{5}\hat{k}$

$\hat{T}'(t) = \frac{9}{5}\cos 3t\,\hat{i} + \frac{9}{5}\sin 3t\,\hat{j}$

$$\boxed{\hat{N}(t) = \cos 3t\,\hat{i} + \sin 3t\,\hat{j}}$$

7) $f(x) = \ln(\cos x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

$$f''(x) = -\sec^2 x$$

$$K(x) = \frac{\sec^2 x}{|1 + \tan^2 x|^{3/2}} = \frac{\sec^2 x}{\sec^3 x} = \cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$K(x) = \cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$.

MAX VALUE OF 1 AT $x = 0$.

8) $\vec{r}(t) = t\hat{i} + t^2\hat{j} + \frac{t^3}{4}\hat{k}$

$\vec{r}'(t) = \hat{i} + 2t\hat{j} + \frac{3t^2}{4}\hat{k}$

$\vec{r}''(t) = 2\hat{j} + \frac{3}{2}t\,\hat{k}$.

$$\|\vec{r}' \times \vec{r}''\| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & \frac{3t^2}{4} \\ 0 & 2 & \frac{3}{2}t \end{vmatrix} \right\| = \sqrt{\left(3t^2 - \frac{3}{2}t^2\right)^2 + \left(\frac{3}{2}t\right)^2 + (2)^2}$$

$$= \sqrt{\frac{9t^4}{4} + \frac{9}{4}t^2 + 4}$$

$$\|\vec{r}'\| = \sqrt{1 + 4t^2 + \frac{9t^4}{16}}$$

$P(2,4,2)$ CORRESPONDS TO $t=2$

$$K(t=2) = \frac{\sqrt{36+9+4}}{\left(\sqrt{1+16+9}\right)^3} = \boxed{\frac{7}{\left(\sqrt{26}\right)^3} \approx 0.0528}$$

9) $\vec{r}(t) = t\,\hat{i} + \ln(\cos t)\,\hat{j} + 5\hat{k}$

$\vec{r}\,'(t) = \hat{i} - \tan t\,\hat{j}$

If $\cos t > 0$, then $\sec t > 0$.

$\|\vec{r}\,'(t)\| = \sqrt{1 + \tan^2 t} = |\sec t| = \sec t$

$\hat{T}(t) = \dfrac{1}{\sec t}\,\hat{i} - \dfrac{\tan t}{\sec t}\,\hat{j}$

$$\boxed{\hat{T}(t) = \cos t\,\hat{i} - \sin t\,\hat{j}}$$

$$\boxed{\hat{N}(t) = -\sin t\,\hat{i} - \cos t\,\hat{j}}$$