MTH 233 Assignment 8 Key

1) $\underbrace{y^2 - x^2 - \sin xy}_{} = 0$

Call this $F(x,y)$. Then $\dfrac{dy}{dx} = \dfrac{-F_x}{F_y}$

$\Downarrow$

$$\frac{dy}{dx} = \frac{-(-2x - y\cos xy)}{2y - x\cos xy}$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}}$$

2) $\underbrace{x^3 + z^2 + ye^{xz} + z\cos y}_{F(x,y,z)} = 0$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2 + yze^{xz})}{2z + xye^{xz} + \cos y}$$

At $(0,0,0)$,

$$\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(e^{xz} - z\sin y)}{2z + xye^{xz} + \cos y}$$

At $(0,0,0)$,

$$\frac{\partial z}{\partial y} = -1$$

3) $\omega = f(x,y)$

$x = r\cos\theta$

$y = r\sin\theta$

a) $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\boxed{\frac{\partial w}{\partial r} = \cos\theta \frac{\partial w}{\partial x} + \sin\theta \frac{\partial w}{\partial y}}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial w}{\partial \theta} = -r\sin\theta \frac{\partial w}{\partial x} + r\cos\theta \frac{\partial w}{\partial y}$$

$$\boxed{\frac{1}{r}\frac{\partial w}{\partial \theta} = -\sin\theta \frac{\partial w}{\partial x} + \cos\theta \frac{\partial w}{\partial y}}$$

b) - USE CRAMER'S RULE TO SOLVE THE LINEAR SYSTEM FOR f_x & f_y.

$$f_x = \frac{\begin{vmatrix} \partial w/\partial r & \sin\theta \\ \frac{1}{r}\partial w/\partial\theta & \cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}} \qquad f_y = \frac{\begin{vmatrix} \cos\theta & \partial w/\partial r \\ -\sin\theta & \frac{1}{r}\partial w/\partial\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}}$$

$$\boxed{f_x = \cos\theta \frac{\partial w}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial w}{\partial \theta}} \qquad \boxed{f_y = \frac{1}{r}\cos\theta \frac{\partial w}{\partial \theta} + \sin\theta \frac{\partial w}{\partial r}}$$

PROBLEM 3 CONTINUED...

c) USING PART (a),

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(f_x \cos\theta + f_y \sin\theta\right)^2$$

$$+ \left(-f_x \sin\theta + f_y \cos\theta\right)^2$$

$$= f_x^2 \cos^2\theta + \cancel{2 f_x f_y \cos\theta \sin\theta} + f_y^2 \sin^2\theta$$

$$+ f_x^2 \sin^2\theta - \cancel{2 f_x f_y \sin\theta \cos\theta} + f_y^2 \cos^2\theta$$

$$= f_x^2 \left(\cos^2\theta + \sin^2\theta\right) + f_y^2 \left(\sin^2\theta + \cos^2\theta\right)$$

$$= f_x^2 + f_y^2 \quad \checkmark$$

4) $g(x,y) = \dfrac{x-y}{xy+2}$, $\vec{v} = 12\hat{i} + 5\hat{j}$, $P(1,-1)$

$$\vec{\nabla} g(x,y) = \frac{(xy+2) - (x-y)(y)}{(xy+2)^2}\hat{i} + \frac{(xy+2)(-1) - (x-y)(x)}{(xy+2)^2}\hat{j}$$

$$\vec{\nabla} g(1,-1) = \frac{1-(2)(-1)}{1^2}\hat{i} + \frac{1(-1) - (2)(1)}{1^2}\hat{j} = 3\hat{i} - 3\hat{j}$$

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{13} = \frac{12\hat{i}}{13} + \frac{5\hat{j}}{13}$$

$$\vec{\nabla} g(1,-1) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{36}{13} - \frac{15}{13} = \boxed{\frac{21}{13}}$$

5) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2} + \ln(xyz)$

$$f_x(x,y,z) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) + \frac{1}{x}$$

$$f_y(x,y,z) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y) + \frac{1}{y}$$

$$f_z(x,y,z) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) + \frac{1}{z}$$

$$\vec{\nabla} f(-1,2,-2) = \left(\frac{1}{27} - 1\right)\hat{i} + \left(-\frac{2}{27} + \frac{1}{2}\right)\hat{j} + \left(\frac{2}{27} - \frac{1}{2}\right)\hat{k}$$

$$\boxed{\vec{\nabla} f(-1,2,-2) = -\frac{26}{27}\hat{i} + \frac{23}{54}\hat{j} - \frac{23}{54}\hat{k}}$$

6) $f(x,y) = \tan^{-1}\left(\frac{\sqrt{x}}{y}\right)$

$$\vec{\nabla} f(x,y) = \frac{1}{1+\frac{x}{y^2}} \cdot \frac{1}{2}\frac{1}{y\sqrt{x}}\,\hat{i} + \frac{1}{1+\frac{x}{y^2}} \cdot \frac{-\sqrt{x}}{y^2}\,\hat{j}$$

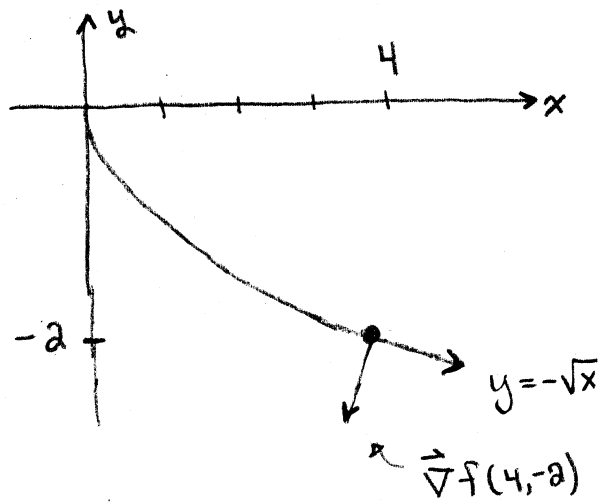
$$\boxed{\vec{\nabla} f(4,-2) = -\frac{1}{16}\hat{i} - \frac{1}{4}\hat{j}}$$

LEVEL CURVE THROUGH $(4,-2)$ IS $\tan^{-1}\left(\frac{\sqrt{x}}{y}\right) = \tan^{-1}(-1)$

OR

$\frac{\sqrt{x}}{y} = -1$ OR

$y = -\sqrt{x}$



7) $V(x,y,z) = 5x^2 - 3xy + xyz$

$$\vec{\nabla} V(x,y,z) = (10x - 3y + yz)\,\hat{i} + (-3x + xz)\,\hat{j} + xy\,\hat{k}$$

$$\vec{\nabla} V(3,4,5) = 38\hat{i} + 6\hat{j} + 12\hat{k}$$

$$\sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624}$$

$$\frac{38\hat{i} + 6\hat{j} + 12\hat{k}}{\sqrt{1624}}$$

8) $\underbrace{x^2-8xyz+y^2+6z^2}_{F(x,y,z)}=0$

$F(1,1,1)=0$

$\vec{\nabla}F(1,1,1)$ IS NORMAL TO SURFACE AT $(1,1,1)$.

$$\vec{\nabla}F(x,y,z)=(2x-8yz)\hat{i}+(-8xz+2y)\hat{j}+(-8xy+12z)\hat{k}$$

$$\vec{n}=\vec{\nabla}F(1,1,1)=-6\hat{i}-6\hat{j}+4\hat{k}$$

NORMAL LINE:

$$\boxed{\begin{aligned} x&=-6t+1\\ y&=-6t+1\\ z&=4t+1\end{aligned}}$$

9) $\underbrace{\sin(xz)-4\cos(yz)}_{F(x,y,z)}=0$

$F(\pi,\pi/2,1)=0$

$\vec{\nabla}F(\pi,\pi/2,1)$ IS NORMAL TO SURFACE AT $(\pi,\pi/2,1)$.

$$\vec{\nabla}F(x,y,z)=z\cos(xz)\hat{i}+4z\sin(yz)\hat{j}+\left(x\cos(xz)+4y\sin(yz)\right)\hat{k}$$

$$\vec{n}=\vec{\nabla}F(\pi,\pi/2,1)=-\hat{i}+4\hat{j}+\pi\hat{k}$$

TANGENT PLANE:

$$\boxed{\begin{gathered}-1(x-\pi)+4\left(y-\tfrac{\pi}{2}\right)+\pi(z-1)=0\\ \text{OR}\\ -x+4y+\pi z=2\pi\end{gathered}}$$

10) $G(x,y,z) = \frac{x}{z} + \frac{z}{y^2}$

$$\vec{\nabla} G(x,y,z) = \frac{1}{z}\hat{i} - \frac{2z}{y^3}\hat{j} + \left(-\frac{x}{z^2} + \frac{1}{y^2}\right)\hat{k}$$

$$\vec{\nabla} G(1,2,-2) = -\frac{1}{2} + \frac{1}{2}\hat{j} + 0\hat{k}$$

MOST RAPID DECREASE IN DIRECTION OPPOSITE GRADIENT.

$$\frac{-\vec{\nabla} G(1,2,-2)}{\|\vec{\nabla} G(1,2,-2)\|} = \boxed{\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}}$$

RATE OF DECREASE $= -\|\vec{\nabla} G(1,2,-2)\|$

$$= -\frac{1}{\sqrt{2}}$$