MTH 233 Assignment 9 key

1) $f(x,y) = 15x^3 - 3xy + 15y^3$

$f_x(x,y) = 45x^2 - 3y \overset{\cdot}{=} 0 \Rightarrow y = 15x^2$

$f_y(x,y) = -3x + 45y^2 = 0$

$-3x + 45\left(15x^2\right)^2 = 0$

$10125x^4 - 3x = 0$

$3x\left(3375x^3 - 1\right) = 0$

$x = 0$ or $x = \dfrac{1}{\sqrt[3]{3375}} = \dfrac{1}{15}$

$\Downarrow \qquad\qquad \Downarrow$

$y = 0 \qquad\qquad y = \dfrac{1}{15}$

CRIT PTS ARE

$(0,0)$ AND $\left(\frac{1}{15}, \frac{1}{15}\right)$

2) $f(x,y) = 9 - x^4y^4$

$f_x(x,y) = -4x^3y^4 = 0$

$f_y(x,y) = -4x^4y^3 = 0$

$\Rightarrow$

CRIT PTS

$x = 0$, y ANY NUMBER $(0,y)$

OR

$y = 0$, x AND NUMBER $(x,0)$

$$D = \begin{vmatrix} -12x^2y^4 & -16x^3y^3 \\ -16x^3y^3 & -12x^4y^2 \end{vmatrix}$$

IF $x = 0$ -OR- $y = 0$,
THEN $D = 0$.

$D = 0$ FOR EVERY CRIT PT.

BUT, NOTICE THAT $f(x,y)$ MUST HAVE A MAX VALUE OF 9 SINCE

$f(x,y) = 9 -$ SOMETHING NONEGATIVE.

$\Downarrow$

ALL CRIT PTS GIVE A MAX VALUE (ABSOLUTE & RELATIVE):

$f(0,y) = 9$

$f(x,0) = 9$.

3) $g(x,y) = x^2 + x - 3xy + y^3 - 5$

$g_x(x,y) = 2x + 1 - 3y = 0 \Rightarrow 2y^2 + 1 - 3y = 0$

$g_y(x,y) = -3x + 3y^2 = 0 \longrightarrow$

$x = y^2$

$2y^2 - 3y + 1 = 0$

$(2y-1)(y-1) = 0$

$\downarrow \qquad \downarrow$

$y = \frac{1}{2} \qquad y = 1$

$x = \frac{1}{4} \qquad x = 1$

CRIT PTS ARE $\left(\frac{1}{4}, \frac{1}{2}\right)$ & $(1,1)$.

$$D = \begin{vmatrix} 2 & -3 \\ -3 & 6y \end{vmatrix} = 12y - 9$$

$D(1,1) = 3$ AND $f_{xx}(1,1) = 2$

$\Rightarrow$ $\boxed{f(1,1) = -5 \text{ IS A REL MIN.}}$

$D\left(\frac{1}{4}, \frac{1}{2}\right) = -3 \Rightarrow$ $\boxed{\left(\frac{1}{4}, \frac{1}{2}, \frac{-79}{16}\right) \text{ IS A SADDLE PT.}}$

4) $f(x,y) = x^3 - 2xy + xy^2 - 7$

$f_x(x,y) = 3x^2 - 2y + y^2 = 0$

$f_y(x,y) = -2x + 2xy = 0 \Rightarrow 2x(y-1) = 0$

$x = 0$

$y^2 - 2y = 0$

$y(y-2) = 0$

$y = 0,\ y = 2$

$y = 1$

$3x^2 = 1$

$x = \pm \frac{1}{\sqrt{3}}$

$$D = \begin{vmatrix} 6x & -2+2y \\ -2+2y & 2x \end{vmatrix} = 12x^2 - (2y-2)^2$$

CRIT PTS ARE

$(0,0),\ (0,2),\ \left(\frac{1}{\sqrt{3}}, 1\right),\ \left(-\frac{1}{\sqrt{3}}, 1\right)$

$D(0,0) = -4 \Rightarrow$ $(0,0,-7)$ IS A SADDLE PT

$D(0,2) = -4 \Rightarrow$ $(0,2,-7)$ IS A SADDLE PT.

$D\left(\frac{1}{\sqrt{3}}, 1\right) = 4$ AND $f_{xx}\left(\frac{1}{\sqrt{3}}, 1\right) = \frac{6}{\sqrt{3}}$ (POS) $\Rightarrow$ $f\left(\frac{1}{\sqrt{3}}, 1\right) = -7 - \frac{2\sqrt{3}}{9}$ IS A REL MIN.

$D\left(-\frac{1}{\sqrt{3}}, 1\right) = 4$ AND $f_{xx}\left(-\frac{1}{\sqrt{3}}, 1\right) = -\frac{6}{\sqrt{3}}$ (NEG) $\Rightarrow$ $f\left(-\frac{1}{\sqrt{3}}, 1\right) = -7 + \frac{2\sqrt{3}}{9}$ IS A REL MAX

5) THIS PROBLEM IS A LOT LIKE #2.

THE VALUES OF $f(x,y) = x^2y^2$ ARE ALWAYS NONNEGATIVE -- AT LEAST ZERO! AT $(0,b)$ OR $(a,0)$,

f TAKES ON THAT ZERO MINIMUN VALUES.

6) $f(x,y,z) = x^2 - y + yz \Rightarrow \vec{\nabla} f(x,y,z) = 2x\,\hat{i} + (-1+z)\hat{j} + y\hat{k}$

CONSTRAINT : $x + y - z^2 = 0 \Rightarrow \vec{\nabla} g(x,y,z) = \hat{i} + \hat{j} - 2z\hat{k}$

a)
$$\boxed{\begin{aligned} 2x &= \lambda \\ -1+z &= \lambda \\ y &= -2z\lambda \\ x+y-z^2 &= 0 \end{aligned}}$$

b) $f\left(-\frac{1}{3}, \frac{4}{9}, \frac{1}{3}\right) = -\frac{5}{27} \approx -0.1852$ = MAX VALUE

$f\left(-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right) = -\frac{3}{16} = -0.1875$ = MIN VALUE

c) $\lambda = -\frac{2}{3}$ FOR THE FIRST PT

$\lambda = -\frac{1}{2}$ FOR THE SECOND PT

7) $f(x,y) = y^2 - 4x$ SUBJECT TO $x^2 + y^2 = 9$

$-4 = \lambda 2x$

$2y = \lambda 2y \Rightarrow 2y - \lambda 2y = 2y(1-\lambda) = 0$

$x^2 + y^2 = 9$

$y = 0, \quad \lambda = 1$

$x = \pm 3 \qquad x = -2$

$\Downarrow$

$y = \pm\sqrt{5}$

PTS TO CHECK ARE

$(3,0), (-3,0), (-2,\sqrt{5}), (-2,-\sqrt{5})$

$f(3,0) = -12$

$f(-3,0) = 12$

$f(-2,\sqrt{5}) = 13$

$f(-2,-\sqrt{5}) = 13$

MAX VALUE IS 13 AT $(-2,\pm\sqrt{5})$.

MIN VALUE IS -12 AT $(3,0)$.

8) $f(x,y) = x^2 y$ SUBJECT TO $x^2 + y^2 = 1$

$$2xy = \lambda 2x \longrightarrow 2xy - \lambda 2x = 0$$

$$x^2 = \lambda 2y \qquad 2x(y-\lambda) = 0$$

$$x^2 + y^2 = 1$$

$x = 0 \Downarrow y = \pm 1$

$y = \lambda \Downarrow x^2 = 2\lambda^2 \Downarrow 2\lambda^2 + \lambda^2 = 1$

$\lambda^2 = \frac{1}{3}$

$\lambda = \pm \frac{1}{\sqrt{3}}$

$\Downarrow$

$y = \pm \frac{1}{\sqrt{3}}$

$\Rightarrow x = \pm \frac{\sqrt{2}}{\sqrt{3}}$ OR $\mp \frac{\sqrt{2}}{\sqrt{3}}$

PTS TO CONSIDER:

$(0,1)$, $(0,-1)$, $\left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$,

$\left(-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$,

$\left(-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

$f(0,\pm 1) = 0$

$f\left(\pm\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}} \longleftarrow$ MAX VALUE.

$f\left(\pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}} \longleftarrow$ MIN VALUE