Math 233 - Test 1
February 8, 2024

Name Key

Score ________

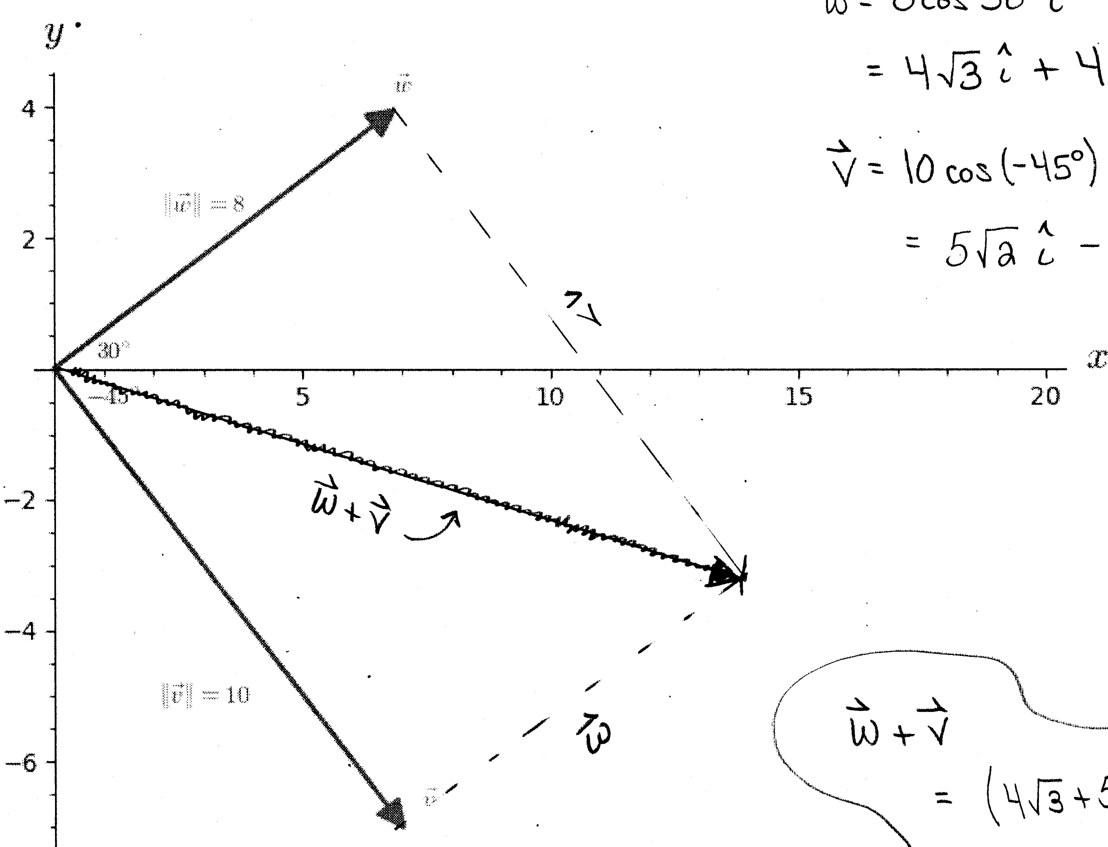
Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Determine a vector in 2-dimensional space that is perpendicular to $\vec{u} = 3\hat{\imath} - 5\hat{\jmath}$ and has magnitude 4.

LET $\vec{v} = 5\hat{\imath} + 3\hat{\jmath}$. NOW THE SLOPES OF $\vec{u}$ & $\vec{v}$ ARE OPPOSITE RECIPROCALS.

NOW LET'S DO $\frac{4\vec{v}}{\|\vec{v}\|}$: $\vec{w} = \frac{4\vec{v}}{\sqrt{25+9}} = \frac{4}{\sqrt{34}}\vec{v} = \boxed{\frac{20}{\sqrt{34}}\hat{\imath} + \frac{12}{\sqrt{34}}\hat{\jmath}}$

2. (8 points) Referring to the figure below, sketch $\vec{w} + \vec{v}$. Then find the component form of $\vec{w} + \vec{v}$.



$\vec{w} = 8\cos 30^\circ\,\hat{\imath} + 8\sin 30^\circ\,\hat{\jmath}$

$= 4\sqrt{3}\,\hat{\imath} + 4\,\hat{\jmath}$

$\vec{v} = 10\cos(-45^\circ)\,\hat{\imath} + 10\sin(-45^\circ)\,\hat{\jmath}$

$= 5\sqrt{2}\,\hat{\imath} - 5\sqrt{2}\,\hat{\jmath}$

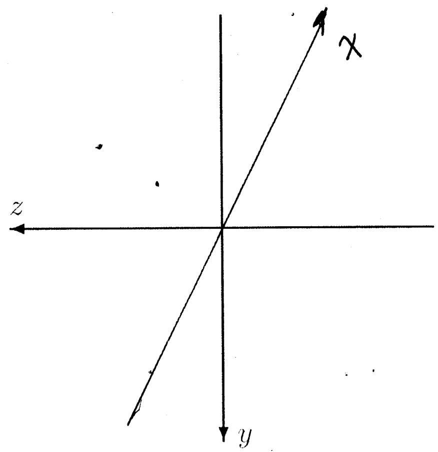
$\vec{w} + \vec{v}$

$= (4\sqrt{3} + 5\sqrt{2})\,\hat{\imath}$

$+ (4 - 5\sqrt{2})\,\hat{\jmath}$

$\approx 13.999\,\hat{\imath} - 3.071\,\hat{\jmath}$

3. (3 points) A 3-dimensional rectangular coordinate system is set up in such a way that the positive y and z axes are as indicated below. Describe the direction of the positive x-axis, and say how you know.



THE POS. X-AXIS WOULD POINT INTO THE PAPER BY THE RIGHT-HAND RULE.

4. (10 points) Consider the line segment from $P(3, -2, 1)$ to $Q(6, -6, -8)$.

(a) Find the midpoint of the line segment.

$$m = \left(\frac{3+6}{2}, \frac{-2+-6}{2}, \frac{1+-8}{2}\right) = \boxed{\left(\frac{9}{2}, -4, \frac{-7}{2}\right)}$$

(b) Find the length of the segment.

$$\sqrt{(6-3)^2+(-6+2)^2+(-8-1)^2} = \sqrt{9+16+81} = \boxed{\sqrt{106}}$$

(c) Find a set of parametric equations for the line **segment**.

$\vec{v} = \overrightarrow{PQ} = 3\hat{i} - 4\hat{j} - 9\hat{k}$

using $P(3,-2,1)$ $\longrightarrow$

$$\boxed{\begin{aligned} x &= 3+3t \\ y &= -2-4t \\ z &= 1-9t \end{aligned} \qquad 0 \le t \le 1}$$

(d) Find symmetric equations for the line through P and Q.

SOLVING FOR t.

$$\boxed{\frac{x-3}{3} = \frac{y+2}{-4} = \frac{z-1}{-9}}$$

5. (8 points) Let $\vec{x} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{y} = 3\hat{\imath} + 8\hat{\jmath} - 4\hat{k}$.

(a) Let $\vec{w}$ be the projection of $\vec{y}$ onto $\vec{x}$. Compute $\vec{w}$.

$$\vec{w} = \text{proj}_{\vec{x}}\, \vec{y} = \frac{\vec{x}\cdot\vec{y}}{\vec{x}\cdot\vec{x}}\vec{x} = \frac{18}{6}\vec{x} = \boxed{3\vec{x} = 6\hat{\imath} + 3\hat{\jmath} - 3\hat{k}}$$

(b) Let $\vec{z} = \vec{y} - \vec{w}$ and compute $\vec{z}$.

$$\vec{z} = \vec{y} - \vec{w} = (3\hat{\imath} + 8\hat{\jmath} - 4\hat{k}) - (6\hat{\imath} + 3\hat{\jmath} - 3\hat{k}) = \boxed{-3\hat{\imath} + 5\hat{\jmath} - \hat{k}}$$

(c) Now compute $\vec{w}\cdot\vec{z}$. What does your answer say about $\vec{w}$ and $\vec{z}$?

$$\vec{w}\cdot\vec{z} = 6(-3) + 3(5) + (-3)(-1) = \boxed{0} \Rightarrow$$

$\vec{w}$ & $\vec{z}$ ARE ORTHOGONAL.

6. (8 points) Find a vector of magnitude 10 that is orthogonal to both $\vec{a} = \langle 2, 4, -1, \rangle$ and $\vec{b} = \langle 1, 2, 1\rangle$.

$\vec{u} =$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 4 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \hat{\imath}(4+2) - \hat{\jmath}(2+1) + \hat{k}(4-4)$$

$$= 6\hat{\imath} - 3\hat{\jmath}$$

$$\|\vec{a}\times\vec{b}\| = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$$

$$\frac{10\vec{u}}{3\sqrt{5}} = \frac{20}{\sqrt{5}}\hat{\imath} - \frac{10}{\sqrt{5}}\hat{\jmath}$$

$$= \boxed{4\sqrt{5}\,\hat{\imath} - 2\sqrt{5}\,\hat{\jmath}}$$

7. (6 points) Let $\vec{w} = \frac{2}{11}\hat{\imath} + \frac{6}{11}\hat{\jmath} + \frac{9}{11}\hat{k}$.

(a) Confirm that $\vec{w}$ is a unit vector.

$$\|\vec{w}\| = \sqrt{\left(\frac{2}{11}\right)^2 + \left(\frac{6}{11}\right)^2 + \left(\frac{9}{11}\right)^2} = \sqrt{\frac{4+36+81}{121}} = \sqrt{\frac{121}{121}} = 1 \checkmark$$

(b) Show that each component of $\vec{w}$ is the cosine of the angle between $\vec{w}$ and the corresponding coordinate axis.

$$\cos\alpha = \frac{\vec{w}\cdot\hat{\imath}}{\|\vec{w}\|\,\|\hat{\imath}\|} = \vec{w}\cdot\hat{\imath} = \frac{2}{11} \checkmark$$

$$\cos\beta = \frac{\vec{w}\cdot\hat{\jmath}}{\|\vec{w}\|\,\|\hat{\jmath}\|} = \vec{w}\cdot\hat{\jmath} = \frac{6}{11} \checkmark$$

$$\cos\gamma = \frac{\vec{w}\cdot\hat{k}}{\|\vec{w}\|\,\|\hat{k}\|} = \vec{w}\cdot\hat{k} = \frac{9}{11} \checkmark$$

8. (16 points) Consider the points $A(3,5,-7)$, $B(6,-1,3)$, and $C(-2,-2,8)$.

(a) Show that the points are NOT collinear.

$\vec{AB} = 3\hat{i} - 6\hat{j} + 10\hat{k}$

$\vec{AC} = -5\hat{i} - 7\hat{j} + 15\hat{k}$

$\vec{AB}$ IS NOT A SCALAR MULTIPLE OF $\vec{AC}$. $\Rightarrow$ $\vec{AB}$ AND $\vec{AC}$ ARE NOT PARALLEL. $\Rightarrow$ POINTS CANNOT LIE ALONG A LINE.

(b) Find the area of $\triangle ABC$.

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 10 \\ -5 & -7 & 15 \end{vmatrix} = \hat{i}(-20) - \hat{j}(95) + \hat{k}(-51) = -20\hat{i} - 95\hat{j} - 51\hat{k}$$

$$\text{AREA} = \frac{\|\vec{AB} \times \vec{AC}\|}{2} = \frac{\sqrt{12026}}{2} \approx \boxed{54.83\ u^2}$$

(c) Find an equation of the plane containing the points A, B, and C.

$\vec{n} = 20\hat{i} + 95\hat{j} + 51\hat{k}$

Using $(-2,-2,8)$ ...

$20(x+2) + 95(y+2) + 51(z-8) = 0$

OR

$\boxed{20x + 95y + 51z = 178}$

(d) Find the measure of the angle between your plane in part (c) and the plane $2x - 7y + 6z = 0$. (If you were unable to do part (c), just thoroughly explain how you would find this angle.)

$\vec{n}_1 = 20\hat{i} + 95\hat{j} + 51\hat{k}$

$\vec{n}_2 = 2\hat{i} - 7\hat{j} + 6\hat{k}$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\,\|\vec{n}_2\|}$$

$$\cos\theta = \frac{319}{\sqrt{12026}\sqrt{89}} \approx 0.308\ldots$$

$\boxed{\theta \approx 72.04^\circ}$

9. (5 points) Find a vector-valued function whose graph is the line described by

$$x = 2y - 6 = \frac{7-z}{3}.$$

$x = t$

$y = \frac{1}{2}t + 3$

$z = -3t + 7$

$$\boxed{\vec{r}(t) = t\hat{i} + \left(\tfrac{1}{2}t + 3\right)\hat{j} + (-3t+7)\hat{k}}$$

10. (8 points) The distance from a line to a point not on the line is given by $\frac{\|\vec{PQ} \times \vec{v}\|}{\|\vec{v}\|}$, where $\vec{v}$ is a vector parallel to the line, P is a point on the line, and Q is the point not on the line.

Find the distance from the line

$$x = 2t + 7, \qquad y = -t - 3, \qquad z = 9$$

to the point $(5, -4, 3)$.

$P\,(7, -3, 9)$

$Q\,(5, -4, 3)$

$\vec{PQ} = -2\hat{\imath} - \hat{\jmath} - 6\hat{k}$

$\vec{v} = 2\hat{\imath} - \hat{\jmath}$

$$\vec{PQ} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & -1 & -6 \\ 2 & -1 & 0 \end{vmatrix} = \hat{\imath}(-6) - \hat{\jmath}(12) + \hat{k}(4) = -6\hat{\imath} - 12\hat{\jmath} + 4\hat{k}$$

$$\|\vec{PQ} \times \vec{v}\| = \sqrt{36 + 144 + 16} = 14$$

$$\|\vec{v}\| = \sqrt{5}$$

$$\boxed{D = \frac{14}{\sqrt{5}}}$$

11. (8 points) Let $\vec{r}(t) = (\frac{1}{2}t + 1)\,\hat{\imath} + (3 - \frac{1}{4}t^2)\,\hat{\jmath}$.

(a) Write the set of parametric equations whose graph is that of $\vec{r}(t)$ Then eliminate the parameter t to obtain an equation in the rectangular coordinates x and y.

$x = \frac{1}{2}t + 1 \Rightarrow t = 2(x-1)$

$y = 3 - \frac{1}{4}t^2$

$$y = 3 - \frac{1}{4}[2(x-1)]^2 \Rightarrow \boxed{y = 3 - (x-1)^2}$$

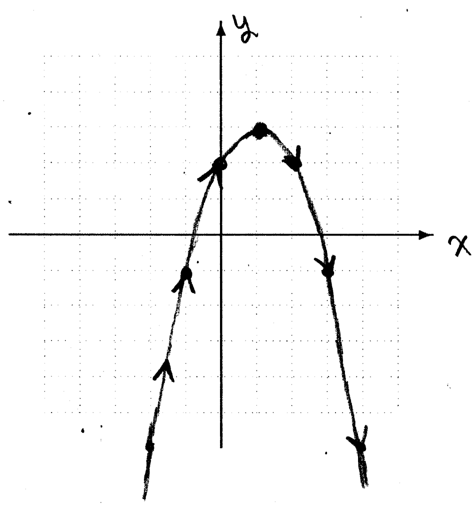
$y = x^2$ Flipped upside down and shifted to vertex at $(1,3)$

(b) Carefully sketch the graph of $\vec{r}(t)$ and draw arrows to show the graph's orientation.

$x = \frac{1}{2}t + 1$

$y = 3 - \frac{1}{4}t^2$

X increases with t



12. (8 points) Let $\vec{r}(t) = \sin 2t\,\hat{\imath} + \cos 2t\,\hat{\jmath} + t\,\hat{k}$.

(a) Describe the graph of $\vec{r}(t)$.

THE GRAPH IS A CIRCULAR HELIX CENTERED ON THE Z-AXIS.

(b) Compute $\|\vec{r}(t)\|$.

$$\|\vec{r}(t)\| = \sqrt{\sin^2 2t + \cos^2 2t + t^2} = \boxed{\sqrt{1+t^2}}$$

(c) Find $\vec{r}\,'(t)$.

$$\boxed{\vec{r}\,'(t) = 2\cos 2t\,\hat{\imath} - 2\sin 2t\,\hat{\jmath} + \hat{k}}$$

(d) Show that $\vec{r}(t)$ is orthogonal to $\vec{r}\,'(t)$ only when $t = 0$.

$$\vec{r}(t)\cdot\vec{r}\,'(t) = (\sin 2t)(2\cos 2t) + (\cos 2t)(-2\sin 2t) + (t)(1)$$

$$= t = 0 \text{ WHEN } t \text{ IS ZERO.}$$

13. (6 points) Let $\vec{r}(t) = \dfrac{t-2}{t^2-4}\,\hat{\imath} + \dfrac{\sin t}{t}\,\hat{\jmath} + |t|\,\hat{k}$.

(a) For which values of t is $\vec{r}(t)$ discontinuous?

$t = 2,\ t = -2,\ t = 0$ (DIVISION BY ZERO IS NOT DEFINED!)

(b) Which of those discontinuities are removable? Briefly say how you know.

$\lim_{t\to 2} \vec{r}(t) = \frac{1}{4}\hat{\imath} + \frac{\sin 2}{2}\hat{\jmath} + 2\hat{k} \longrightarrow$ LIMIT EXISTS AT $t=2$. REMOVABLE.

$\lim_{t\to 0} \vec{r}(t) = -\frac{1}{2}\hat{\imath} + \hat{\jmath} \longrightarrow$ LIMIT EXISTS AT $t=0$. REMOVABLE.

NO LIMIT AT $t=-2$. NONREMOVABLE.