Math 233 - Test 2

March 7, 2024

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (9 points) Let $\vec{r}(t) = \left(\frac{1}{t} + t\right)\hat{\imath} + \left(\frac{1}{t} - t\right)\hat{\jmath}$.

   (a) Show that $\|\vec{r}\,'(t)\| = \frac{\sqrt{2t^4+2}}{t^2}$.

$$\vec{r}\,'(t) = \left(-\frac{1}{t^2}+1\right)\hat{\imath} + \left(-\frac{1}{t^2}-1\right)\hat{\jmath} = \left(\frac{t^2-1}{t^2}\right)\hat{\imath} - \left(\frac{t^2+1}{t^2}\right)\hat{\jmath}$$

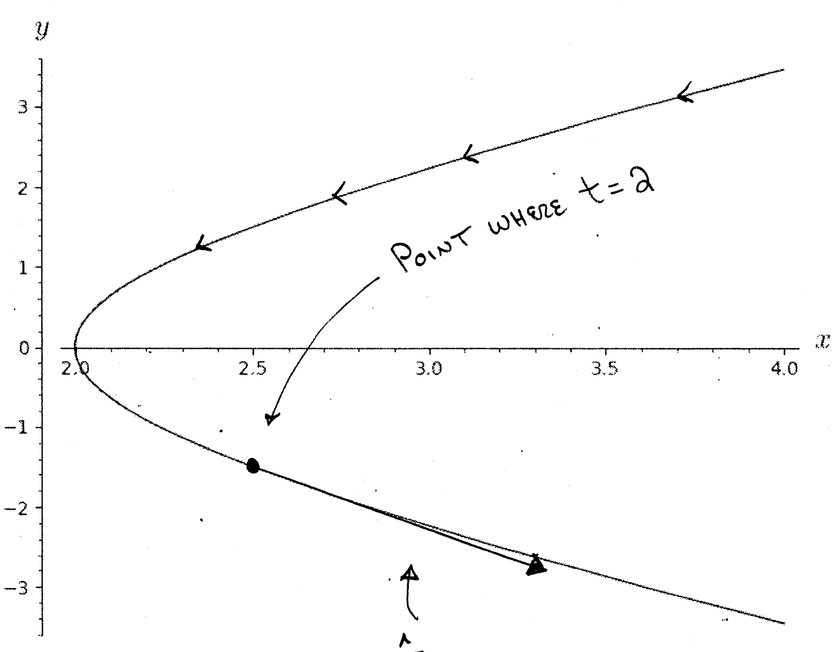
$$\|\vec{r}\,'(t)\| = \frac{\sqrt{(t^2-1)^2+(t^2+1)^2}}{t^2} = \frac{\sqrt{2t^4+2}}{t^2} \checkmark$$

   (b) Find $\hat{T}(2)$.

$$\hat{T}(2) = \frac{\vec{r}\,'(2)}{\|r'(2)\|} = \frac{3\hat{\imath} - 5\hat{\jmath}}{\sqrt{34}} = \boxed{\frac{3}{\sqrt{34}}\hat{\imath} - \frac{5}{\sqrt{34}}\hat{\jmath}}$$

   (c) The graph of r is shown below. Sketch $\hat{T}(2)$ at the point where $t = 2$.

$\vec{r}(2) = 2.5\hat{\imath} - 1.5\hat{\jmath}$



$\hat{T}(2)$ = UNIT VECTOR, TANGENT TO CURVE, IN DIRECTION OF MOTION

2. (6 points) Find $\vec{r}(t)$ given the following information.

$$\vec{r}\,'(t) = \frac{3}{2}(t+1)^{1/2}\,\hat{\imath} + e^{-t}\,\hat{\jmath} + \frac{1}{t+1}\,\hat{k}, \quad \vec{r}(0) = \hat{k}$$

$$\vec{r}(t) = \int \left(\frac{3}{2}(t+1)^{1/2}\,\hat{\imath} + e^{-t}\,\hat{\jmath} + \frac{1}{t+1}\,\hat{k} \right) dt = \left[(t+1)^{3/2} + c_1\right]\hat{\imath} + \left(-e^{-t} + c_2\right)\hat{\jmath} + \left[\ln|t+1| + c_3\right]\hat{k}$$

$$\vec{r}(0) = \hat{k} \Rightarrow 1 + c_1 = 0$$

$$-1 + c_2 = 0$$

$$c_3 = 1$$

$$\boxed{\vec{r}(t) = \left[(t+1)^{3/2} - 1\right]\hat{\imath} + \left(-e^{-t} + 1\right)\hat{\jmath} + \left[\ln|t+1| + 1\right]\hat{k}}$$

3. (8 points) Find the length of the graph of

$$\vec{r}(t) = (1 - t^2)\,\hat{\imath} + \sqrt{8}\,t^2\,\hat{\jmath} + (7 + 2t^3)\,\hat{k}$$

from $t = 0$ to $t = 1$. (Do not use your calculator.)

$$\vec{r}\,'(t) = -2t\,\hat{\imath} + 2\sqrt{8}\,t\,\hat{\jmath} + 6t^2\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 32t^2 + 36t^4}$$

$$= \sqrt{36t^2 + 36t^4}$$

$$= 6t\sqrt{1+t^2}$$

$$\text{ARC LENGTH} = \int_0^1 6t\sqrt{1+t^2}\,dt$$

$$u = 1 + t^2, \quad du = 2t\,dt$$

$$= \int_1^2 3u^{1/2}\,du = 2u^{3/2}\Big|_1^2$$

$$= \boxed{2\sqrt{8} - 2 \approx 3.66}$$

4. (8 points) Let $\vec{r}(t) = (6\sin 2t)\,\hat{\imath} + 5t\,\hat{\jmath} + (6\cos 2t)\hat{k}$. Starting from $t = 0$, find the arc-length parameter $s(t)$, and then reparameterize $\vec{r}$ in terms of s.

$$\vec{r}\,'(t) = 12\cos 2t\,\hat{\imath} + 5\hat{\jmath} - 12\sin 2t\,\hat{\jmath}$$

$$\|\vec{r}\,'(t)\| = \sqrt{144\cos^2 2t + 25 + 144\sin^2 2t}$$

$$= \sqrt{169} = 13$$

$$s = \int_0^t 13\,d\tau = 13t$$

$$t = \frac{s}{13}$$

$$\boxed{\vec{R}(s) = 6\sin\frac{2s}{13}\,\hat{\imath} + \frac{5s}{13}\,\hat{\jmath} + 6\cos\frac{2s}{13}\,\hat{k}}$$

5. (12 points) A volleyball is hit when it is 4 ft above the ground and 12 ft from a 6-ft-high net. It leaves the point of impact with an initial speed of 35 ft/sec at an angle of 27°.

(a) Find the vector-valued function that gives the position of the volleyball at time t. (Use $g = 32$ ft/sec^2.)

$$\boxed{\vec{r}(t) = 35\cos 27°\, t\ \hat{\imath} + \left(-16t^2 + 35\sin 27°\, t + 4\right)\hat{\jmath}}$$

(b) When does the volleyball reach its maximum height, and what is the maximum height?

$$-32t + 35\sin 27° = 0$$

$$\Downarrow$$

$$t = \frac{35\sin 27°}{32} \approx 0.49655 \text{ SECONDS}$$

USING THIS t

$$-16t^2 + 35\sin 27°\, t + 4$$

$$\approx \boxed{7.945 \text{ FT}}$$

(c) How high is the volleyball when it passes over the net?

$$35\cos 27°\, t = 12$$

$$\Downarrow$$

$$t = \frac{12}{35\cos 27°}$$

$$\approx 0.38480 \text{ SECONDS}$$

USING THIS t

$$-16t^2 + 35\sin 27°\, t + 4$$

$$\approx \boxed{7.745 \text{ FT}}$$

(d) Find the range and the flight time of the volleyball.

$$-16t^2 + 35\sin 27°\, t + 4 = 0$$

$$\Downarrow$$

$$t = \frac{-35\sin 27° - \sqrt{(35\sin 27°)^2 + 256}}{-32}$$

$$\approx \boxed{1.20123 \text{ SECONDS}}$$

USING THIS t

$$35\cos 27°\, t$$

$$= \boxed{37.460 \text{ FT}}$$

6. (8 points) Compute the curvature of the graph of $\vec{r}(t) = \frac{t^3}{3}\hat{\imath} + \frac{t^2}{2}\hat{\jmath} + t\hat{k}$ at the point $(1/3, 1/2, 1)$.

POINT WHERE $t = 1$

$\vec{r}'(t) = t^2\hat{i} + t\hat{j} + \hat{k}$

$\vec{r}''(t) = 2t\hat{i} + \hat{j}$

$\vec{r}'(1) = \hat{i} + \hat{j} + \hat{k}$

$\|\vec{r}'(1)\| = \sqrt{3}$

$\vec{r}''(1) = 2\hat{i} + \hat{j}$

$$\vec{r}'(1) \times \vec{r}''(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k}$$

$\|\vec{r}'(1) \times \vec{r}''(1)\| = \sqrt{6}$

$$K(t=1) = \frac{\sqrt{6}}{(\sqrt{3})^3} = \boxed{\frac{\sqrt{2}}{3}} \approx 0.47140$$

7. (4 points) Look back at the graph of $\vec{r}$ in problem #1. By using the graph of $\vec{r}$, determine if there are any points at which $\hat{N}(t)$ is not defined. Explain.

NO, WE'D BE LOOKING FOR WHERE THE DIRECTION "INTO THE CURVE" CHANGES (INFLECTION PTS). THERE ARE NO SUCH PTS.

8. (2 points) Suppose r is a curve for which $\hat{N}(t)$ exists for every t. Is it possible that $\hat{N}(t_0) = \vec{0}$ for some number t_0? Explain.

Nope! IT'S A UNIT VECTOR

ITS MAGNITUDE IS 1 (NOT EVER 0)

WHENEVER IT EXISTS.

9. (3 points) What are the three important characteristics of the principal unit tangent vector?

1. IT IS A UNIT VECTOR: $\|\hat{T}\| = 1$
2. IT POINTS IN THE DIRECTION OF MOTION ALONG THE CURVE.
3. IT IS TANGENT TO THE CURVE AT EACH PT.

10. (12 points) Each of these equations defines a surface in 3-space. Briefly describe each surface. If possible, do more than just give the name of each surface.

(a) $3x^2 - 9y^2 - 8z^2 = -5$ $\quad 9y^2 + 8z^2 = 3x^2 + 5$

FIX X: ELLIPSES FOR ALL X

FIX Y: HYPERBOLAS

FIX Z: HYPERBOLAS

HYPERBOLOID OF ONE SHEET. CROSS SECTIONS ARE ELLIPSES CENTERED ON X-AXIS.

(b) $4x^2 + 4y^2 = 1$

CIRCULAR CYLINDER -- GENERATING CURVE IS THE CIRCLE $x^2+y^2=\left(\frac{1}{2}\right)^2$, RULINGS ARE PARALLEL TO Z-AXIS.

(c) $-x^2 - 2y^2 + 3z^2 = 4$ $\quad x^2 + 2y^2 = 3z^2 - 4$.

FIX X: HYPERBOLAS

FIX Y: HYPERBOLAS

FIX Z: ELLIPSES FOR BIG ENOUGH Z

HYPERBOLOID OF TWO SHEETS. CROSS SECTIONS FOR $|z| > \frac{2}{\sqrt{3}}$ ARE ELLIPSES CENTERED ON Z-AXIS.

(d) $(x-2)^2 + (y+3)^2 + z^2 = 16$

SPHERE CENTERED AT $(2,-3,0)$ WITH RADIUS 4.

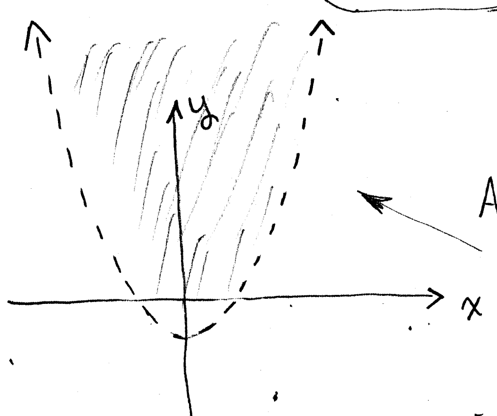
11. (4 points) Find and sketch the domain of the function $g(x, y) = \dfrac{\sin(xy)}{\sqrt{y - x^2 + 1}}$.

MUST HAVE $y - x^2 + 1 > 0$

OR

$y > x^2 - 1$

$$\boxed{\text{DOMAIN} = \{(x,y) : y > x^2 - 1\}}$$



ALL POINTS ABOVE, BUT NOT ON, THE PARABOLA $y = x^2 - 1$.

12. (6 points) Let $f(x, y, z) = \ln(x^2 + y^2 + z^2)$.

(a) What is the domain of f?

MUST HAVE $x^2+y^2+z^2 > 0$ SO ... $\boxed{\{(x,y,z) : (x,y,z) \neq (0,0,0)\}}$

(b) What is the range of f?

$\boxed{\mathbb{R}}$ -- ANY REAL NUMBER IS A POSSIBLE OUTPUT.

(c) Describe the level surface $f(x, y, z) = 5$.

$$\ln(x^2+y^2+z^2) = 5 \Rightarrow x^2+y^2+z^2 = e^5$$

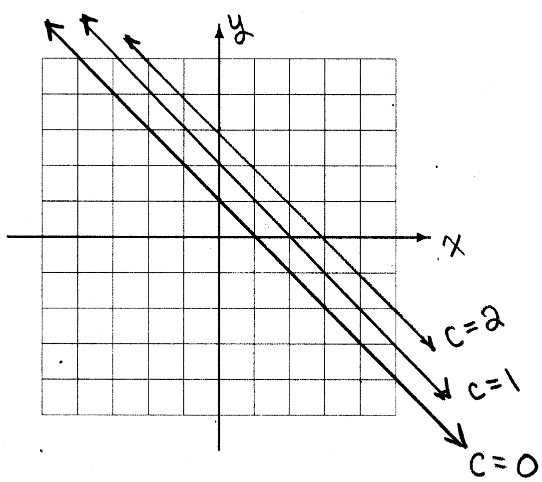
THE LEVEL SURFACE IS THE SPHERE, CENTERED AT $(0,0,0)$, RADIUS $\sqrt{e^5}$

13. (3 points) Let $f(x, y) = x + y - 1$. Sketch the level curves $f(x, y) = c$ at levels $c = 0$, $c = 1$, and $c = 2$. Label the levels.

$c=0$:

$x+y-1=0$

$y=1-x$



$c=1$:

$x+y-1=1$

$y=2-x$

$c=2$:

$x+y-1=2$

$y=3-x$

14. (15 points) Find each limit or show that it does not exist.

(a) $\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2} \cdot \frac{\sqrt{x+y}+2}{\sqrt{x+y}+2}$

$$= \lim_{(x,y)\to(2,2)} \frac{\cancel{(x+y-4)}(\sqrt{x+y}+2)}{\cancel{(x+y-4)}} = \sqrt{2+2}+2$$

$$= \boxed{4}$$

(b) $\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

Along $x=1$:

$$\lim_{y\to -1} \frac{y+1}{1-y^2} = \lim_{y\to -1} \frac{1}{1-y} = \frac{1}{2}$$

Along $y=-1$:

$$\lim_{x\to 1} \frac{1-x}{x^2-1} = \lim_{x\to 1} \frac{-(x-1)}{x^2-1} = \lim_{x\to 1} \frac{-1}{x+1} = -\frac{1}{2}$$

$\boxed{\text{LIMIT DNE.}}$

(c) $\lim_{(x,y)\to(3,3)} \frac{x-y}{x^4-y^4}$

$$= \lim_{(x,y)\to(3,3)} \frac{\cancel{(x-y)}}{\cancel{(x-y)}(x+y)(x^2+y^2)} = \lim_{(x,y)\to(3,3)} \frac{1}{(x+y)(x^2+y^2)}$$

$$= \frac{1}{(3+3)(9+9)} = \boxed{\frac{1}{108}}$$