Math 233 - Test 3

April 11, 2024

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (4 points) Find f_x, f_y, and f_z when $f(x,y,z) = yz\ln(xy^2)$.

$f_x(x,y,z) = \frac{yz}{xy^2}\cdot y^2 = \boxed{\frac{yz}{x}}$

$f_z(x,y,z) = \boxed{y\ln(xy^2)}$

$f_y(x,y,z) = z\ln(xy^2) + \frac{yz}{xy^2}\cdot 2xy = \boxed{z\ln(xy^2) + 2z}$

2. (6 points) Let $f(x,y) = x\cos(2y) + e^{y^3}$.

   (a) Which partial mixed partial derivative is less work to compute, f_{xy} or f_{yx}? Why?

   f_{xy} IS LESS WORK -- THE SECOND TERM IS ZEROED BY THE X-PARTIAL. SEE BELOW.

   (b) Compute whichever mixed partial you said in part (a).

   $f_x(x,y) = \cos 2y$

   $\boxed{f_{xy}(x,y) = -2\sin 2y}$

   (c) In this problem, would you expect $f_{xy} = f_{yx}$? Why or why not?

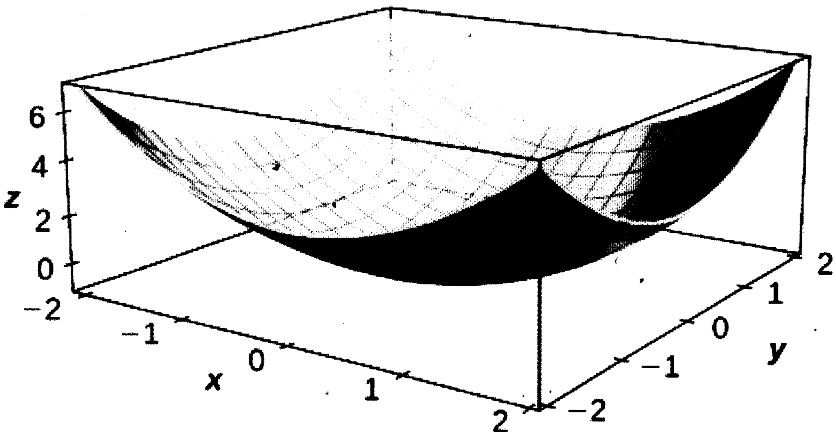
   YES, THEY BOTH WILL BE CONTINUOUS FUNCTIONS ON $\mathbb{R}^2$.

   $\therefore f_{xy} = f_{yx}$

3. (4 points) Look carefully at the graph of $z = f(x, y)$ shown below. Use the graph to determine the sign of $f_x(1, -1)$ and $f_y(1, -1)$. Briefly explain.

AT (1,-1) THE GRAPH SLOPES UP IN THE DIRECTION OF POS. X-AXIS.

$f_x(1,-1)$ IS POSITIVE.



AT (1,-1) THE GRAPH SLOPES DOWN IN THE POS Y-DIRECTION.

$f_y(1,-1)$ IS NEGATIVE.

4. (10 points) Find the linearization of $f(x, y, z) = \dfrac{\sin(xy)}{z}$ at the point $(x, y, z) = (\sqrt{\pi}/2, \sqrt{\pi}/3, \sqrt{\pi})$. Then use your linearization to approximate $f(0.9, 0.6, 1.7)$.

$$f_x(x,y,z) = \frac{y\cos(xy)}{z} = \frac{\sqrt{3}}{6}$$

$$f_y(x,y,z) = \frac{x\cos(xy)}{z} = \frac{\sqrt{3}}{4}$$

$$f_z(x,y,z) = -\frac{\sin(xy)}{z^2} = \frac{-1}{2\pi}$$

AT $\left(\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{3}, \sqrt{\pi}\right)$

$$f\left(\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{3}, \sqrt{\pi}\right) = \frac{1}{2\sqrt{\pi}}$$

$$L(x,y,z) = \frac{1}{2\sqrt{\pi}} + \frac{\sqrt{3}}{6}\left(x - \frac{\sqrt{\pi}}{2}\right) + \frac{\sqrt{3}}{4}\left(y - \frac{\sqrt{\pi}}{3}\right) - \frac{1}{2\pi}\left(z - \sqrt{\pi}\right)$$

$$f(0.9, 0.6, 1.7) \approx L(0.9, 0.6, 1.7) = 0.301578\ldots$$

5. (6 points) Use differentials to approximate the change in $g(x, y) = \tan^{-1}(xy^2)$ as (x, y) changes from $(1, 1)$ to $(1.02, 0.97)$.

$$dz = \frac{1}{1+(xy^2)^2} \cdot y^2\, dx + \frac{1}{1+(xy^2)^2} \cdot 2xy\, dy$$

$$= \frac{y^2}{1+x^2y^4}\, dx + \frac{2xy}{1+x^2y^4}\, dy$$

$$\Delta z \approx \frac{y^2}{1+x^2y^4}\, \Delta x + \frac{2xy}{1+x^2y^4}\, \Delta y$$

$x = y = 1, \quad \Delta x = 0.02, \quad \Delta y = -0.03$

$$\Delta z \approx \frac{1}{2}(0.02) - 0.03 = \boxed{-0.02}$$

6. (4 points) Let $z = x^2(y+1)$. Compute the total differential dz. Near the point $(1,0)$, is z more sensitive to changes in x or to changes in y? Explain.

$$\boxed{dz = 2x(y+1)\,dx + x^2\,dy}$$

AT $(x,y) = (1,0)$,

$$dz = 2\,dx + dy$$

dz IS TWICE AS SENSITIVE TO A CHANGE IN X

7. (4 points) Suppose $v = f(w,x,y,z)$ is differentiable, and w, x, y, and z are differentiable functions of s and t. Write the formula for the derivative of v with respect to t.

$$\boxed{\frac{\partial v}{\partial t} = \frac{\partial v}{\partial w}\cdot\frac{\partial w}{\partial t} + \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial t}}$$

2-4 CHAIN RULE.

8. (4 points) Use partial derivatives to find $\partial z/\partial y$ at the point $(1,1,1)$.

$$\underbrace{z^3 - xy + yz + y^3 - 2}_{F(x,y,z)} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(-x+z+3y^2)}{3z^2+y} = \boxed{\frac{-3}{4} \text{ WHEN } (x,y,z) = (1,1,1)}$$

9. (4 points) The voltage V in a circuit satisfies $V = IR$, where I is the current and R is the resistance. The voltage is dropping as the battery wears out and the resistor heats up. Find a formula for dV/dt, and use it to determine dI/dt when $R = 600$ ohms, $I = 0.04$ amp, $dR/dt = 0.5$ ohm/sec, and $dV/dt = -0.01$ volt/sec.

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt}$$

PLUG IN ...

$$-0.01 = (600)\frac{dI}{dt} + (0.04)(0.5) \Rightarrow \boxed{\frac{dI}{dt} = -5\times10^{-5}\ \text{amp/sec}}$$

10. (8 points) Find a set of parametric equations for the line normal to the surface $z = 2e^{4x^2+2xy-4y}$ at the point $(1, 2, 2)$.

$$F(x,y,z) = 2e^{4x^2+2xy-4y} - z$$

$$\vec{\nabla} F(x,y,z) = 2(8x+2y)e^{4x^2+2xy-4y}\,\hat{\imath} + 2(2x-4)e^{4x^2+2xy-4y}\,\hat{\jmath} - \hat{k}$$

$$\vec{\nabla} F(1,2,2) = 24\hat{\imath} - 4\hat{\jmath} - \hat{k}$$

NORMAL LINE:

$$\boxed{\begin{aligned} x &= 24t + 1 \\ y &= -4t + 2 \\ z &= -t + 2 \end{aligned}}$$

11. (6 points) Find the directional derivative of $g(x, y) = \dfrac{x-y}{xy+2}$ at the point $(1, -1)$ in the direction of $\vec{v} = 12\hat{\imath} + 5\hat{\jmath}$.

$$\vec{\nabla} g(x,y) = \frac{(xy+2)-(x-y)(y)}{(xy+2)^2}\,\hat{\imath} + \frac{(xy+2)(-1)-(x-y)(x)}{(xy+2)^2}\,\hat{\jmath}$$

$$\vec{\nabla} g(1,-1) = \frac{1+2}{1}\,\hat{\imath} + \frac{-1-2}{1}\,\hat{\jmath} = 3\hat{\imath} - 3\hat{\jmath}$$

$$\|\vec{v}\| = \sqrt{144+25} = 13$$

$$\vec{\nabla} g(1,-1) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{3(12)-3(5)}{13} = \boxed{\frac{21}{13}}$$

12. (8 points) Let $G(x, y, z) = \dfrac{x}{z} + \dfrac{z}{y^2}$. Find a unit vector in the direction in which G decreases most rapidly at $P(1, 2, -2)$. What is the corresponding rate of decrease?

$-\vec{\nabla} G(1,2,-2)$

$-\|\vec{\nabla} G(1,2,-2)\|$

$$\boxed{-\frac{1}{\sqrt{2}}}$$

$$\vec{\nabla} G(x,y,z) = \frac{1}{z}\,\hat{\imath} - \frac{2z}{y^3}\,\hat{\jmath} + \left(\frac{1}{y^2} - \frac{x}{z^2}\right)\hat{k}$$

$$\vec{\nabla} G(1,2,-2) = -\frac{1}{2}\,\hat{\imath} + \frac{1}{2}\,\hat{\jmath} + 0\hat{k}$$

$$\boxed{\frac{-\vec{\nabla} G(1,2,-2)}{\|\vec{\nabla} G(1,2,-2)\|} = \frac{1}{\sqrt{2}}\,\hat{\imath} - \frac{1}{\sqrt{2}}\,\hat{\jmath}}$$

13. (12 points) Find the critical points of $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$. Then use the 2nd partials test to classify your critical points.

$f_x(x,y) = -6yx - 6x = 0 \Rightarrow -6x(y+1) = 0 \Rightarrow x = 0$ OR $y = -1$

$f_y(x,y) = 3y^2 - 3x^2 - 6y$

$x = 0$: $3y^2 - 6y = 0$, $3y(y-2) = 0$, $y = 0,\ y = 2$

$y = -1$: $9 - 3x^2 = 0$, $x = \pm\sqrt{3}$

$$D(x,y) = \begin{vmatrix} -6y-6 & -6x \\ -6x & 6y-6 \end{vmatrix} = -36y^2 + 36 - 36x^2$$

CRIT PTS ARE $(0,0)$, $(0,2)$, $(\sqrt{3},-1)$, $(-\sqrt{3},-1)$

$D(0,0) = 36$, $f_{xx}(0,0) = -6 \Rightarrow$ $\boxed{f(0,0) = 1 \text{ IS A REL MAX}}$

$D(0,2) = -108 \Rightarrow$ $\boxed{(0,2,-3) \text{ IS A SADDLE PT}}$

$D(\sqrt{3},-1) = -108 \Rightarrow$ $\boxed{(\sqrt{3},-1,-3) \text{ IS A SADDLE PT}}$

$D(-\sqrt{3},-1) = -108 \Rightarrow$ $\boxed{(-\sqrt{3},-1,-3) \text{ IS A SADDLE PT}}$

14. (4 points) Suppose you are using the 2nd partials test and it is inconclusive. Describe two things that you might do in order to help you draw a conclusion.

1. LOOK AT THE GRAPH (IF YOU HAVE GOOD TECHNOLOGY)

2. TEST POINTS AROUND THE CRIT PT TO SEE HOW THE FUNCTION VALUES CHANGE.

15. (12 points) Use Lagrange multipliers to find the minimum and maximum values of $f(x,y) = 3x + 4y$ on the circle $\underbrace{x^2 + y^2}_{g(x,y)} = 1$.

$\vec{\nabla} f(x,y) = 3\hat{i} + 4\hat{j}$

$\vec{\nabla} g(x,y) = 2x\hat{i} + 2y\hat{j}$

$3 = \lambda 2x \Rightarrow x = \frac{3}{2\lambda}$

$4 = \lambda 2y \Rightarrow y = \frac{4}{2\lambda}$

$x^2 + y^2 = 1$

$\frac{9}{4\lambda^2} + \frac{16}{4\lambda^2} = 1$

$\frac{25}{4\lambda^2} = 1$

$\lambda^2 = \frac{25}{4}$

$\lambda = \pm \frac{5}{2}$

$\lambda = \frac{5}{2} \Rightarrow x = \frac{3}{5},\ y = \frac{4}{5}$

$\lambda = -\frac{5}{2} \Rightarrow x = -\frac{3}{5},\ y = -\frac{4}{5}$

$f\left(\frac{3}{5}, \frac{4}{5}\right) = 5 =$ MAX VALUE

$f\left(-\frac{3}{5}, -\frac{4}{5}\right) = -5 =$ MIN VALUE

16. (4 points) Refer back to the problem above. Explain how we can be certain that the minimum and maximum values exist even before starting work on the problem.

f IS A CONTINUOUS FUNCTION ON $\mathbb{R}^2$ AND THE SET $\{(x,y) : x^2 + y^2 = 1\}$ IS CLOSED & BOUNDED. $\Longrightarrow$ f TAKES ON A MAX & A MIN.

(EXTREME VALUE THM)