Math 233 - Final Exam B

May 9, 2024

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary.

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1. (10 points) A golf ball is hit from the ground toward a vertical cliff that is 150 m away. The ball is launched at a 40° angle with respect to the horizontal, and its initial speed is 70 m/s. At what height will the ball strike the cliff? Will the ball ever reach its maximum possible height? Explain. (Use $g = 9.81\,\text{m/s}^2$.)

$$\vec{r}(t) = 70\cos 40^\circ\, t\ \hat{\imath} + \left(-\frac{9.81}{2}t^2 + 70\sin 40^\circ\, t\right)\hat{\jmath}$$

$$150 = 70\cos 40^\circ\, t$$

$$\Downarrow$$

$$t = \frac{150}{70\cos 40^\circ} \approx 2.7973\text{ s}$$

This t.

$$-\frac{9.81}{2}t^2 + 70\sin 40^\circ\, t$$

$$\approx 87.4838\text{ m}$$

$\boxed{\text{About } 87.5\text{ m}}$

Max height would occur when

$$-9.81t + 70\sin 40^\circ = 0$$

–or–

$$t \approx 4.5867\text{ s}$$

The ball will hit the cliff on its way up, before it reaches a max.

2. (10 points) Find each limit or show that it does not exist.

(a) $\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}$ 0/0

$$\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}\cdot\frac{\sqrt{x-y}+1}{\sqrt{x-y}+1}$$

$$= \lim_{(x,y)\to(2,1)} \frac{\cancel{(x-y-1)}(\sqrt{x-y}+1)}{\cancel{x-y-1}} = \sqrt{2-1}+1$$

$$= \boxed{2}$$

(b) $\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$ 0/0

$x=2$: $\lim_{y\to1} \frac{0}{(y-1)^2} = 0$

LIMIT DNE

$y=1$: $\lim_{x\to2} \frac{0}{(x-2)^2} = 0$

$x=2y$: $\lim_{y\to1} \frac{(2y-2)(y-1)}{(2y-2)^2+(y-1)^2} = \lim_{y\to1} \frac{2(y-1)^2}{5(y-1)^2} = \frac{2}{5}$

3. (10 points) Let $w = xyz$.

(a) Compute the total differential dw.

$$\boxed{dw = yz\,dx + xz\,dy + xy\,dz}$$

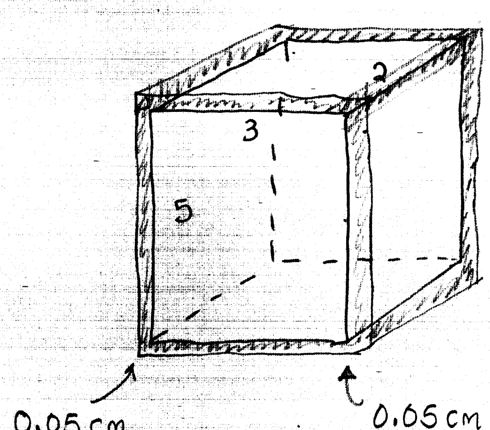
(b) Use differentials to estimate the change in w as (x, y, z) changes from $(5, 3, 2)$ to $(5.1, 3.1, 2.1)$.

$$\Delta w \approx yz\,\Delta x + xz\,\Delta y + xy\,\Delta z$$

WHEN $x = 5,\ y = 3,\ z = 2,\ \Delta x = \Delta y = \Delta z = 0.1 \ldots$

$$\Delta w \approx [6 + 10 + 15](0.1) = \boxed{3.1}$$

(c) Your answer is part (b) is an approximation for the volume of the walls of a empty box with inside dimensions 5 m by 3 m by 2 m, when the walls are 5 cm thick. Explain or illustrate this idea.



VOLUME OF WALLS =

$$(5.1 \times 3.1 \times 2.1) - (5 \times 3 \times 2)$$

$$= 3.201\ m^3$$

4. (10 points) Consider the surface described by the equation $z = 2e^{4x^2+2xy-4y}$.

(a) Find an equation of the plane tangent to the surface at the point $(1, 2, 2)$.

$$F(x,y,z) = 2e^{4x^2+2xy-4y} - z$$

$$\vec{\nabla} F(x,y,z) = (8x+2y)(2e^{4x^2+2xy-4y})\hat{i} + (2x-4)(2e^{4x^2+2xy-4y})\hat{j} - \hat{k}$$

$$\vec{n} = \vec{\nabla} F(1,2,2) = (12)(2)\hat{i} + (-2)(2)\hat{j} - \hat{k}$$

$$= 24\hat{i} - 4\hat{j} - \hat{k}$$

TANGENT PLANE:

$$24(x-1) - 4(y-2) - (z-2) = 0$$

- OR -

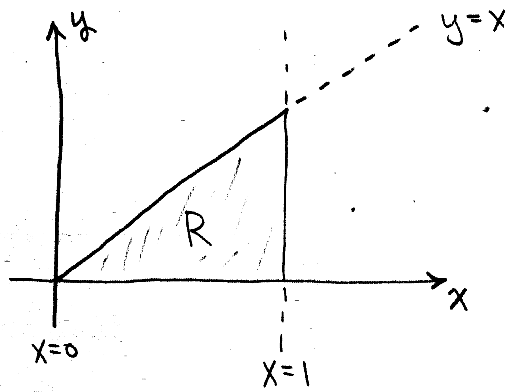
$$24x - 4y - z = 14$$

(b) Find a set of parametric equations for the line normal to the surface at the point $(1, 2, 2)$.

$\vec{n} = 24\hat{i} - 4\hat{j} - \hat{k}$ POINT $(1,2,2)$

$$x = 24t + 1$$
$$y = -4t + 2$$
$$z = -t + 2$$

5. (10 points) Consider the double integral $\iint_R \frac{\sin x}{x}\, dA$, where R is the triangular region in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$. Sketch the region R, and set up the corresponding iterated integrals with both orders of integration. Then choose one of your iterated integrals and evaluate it.

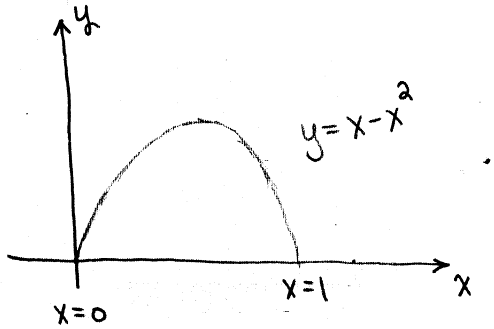


$$\boxed{\int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{\sin x}{x}\, dy\, dx} = \boxed{\int_{y=0}^{y=1} \int_{x=y}^{x=1} \frac{\sin x}{x}\, dx\, dy}$$

$$\int_0^1 y\,\frac{\sin x}{x}\Big|_{y=0}^{y=x} dx = \int_0^1 \sin x\, dx$$

$$= -\cos x \Big|_{x=0}^{x=1} = \boxed{1 - \cos(1) \approx 0.4597}$$

6. (10 points) A region in space lies in the first octant (where $x, y, z \geq 0$) where it is bounded by the cylinder $y = x - x^2$ and the planes $z = 0$ and $z = y$. The volume of the region is $1/60$ units^3. Use a triple integral to find the average value of $f(x, y, z) = 2x^2$ over the region.



Z-AXIS COME OUT OF PAPER.

THE PLANE Z=Y PASSES THROUGH THE X-AXIS (Y=0) AND RISES UP OFF THE PAPER AS Y INCREASES.

THE PLANE Z=0 IS THE PLANE OF THE PAPER.

THE SOLID IS A WEDGE SHAPE IN OCTANT 1.

$$\text{AVERAGE VALUE} = \frac{1}{1/60} \iiint_S 2x^2 \, dV$$

$$= 60 \int_{x=0}^{x=1} \int_{y=0}^{y=x-x^2} \int_{z=0}^{z=y} 2x^2 \, dz \, dy \, dx = 60 \int_0^1 \int_0^{x-x^2} 2x^2 y \, dy \, dx$$

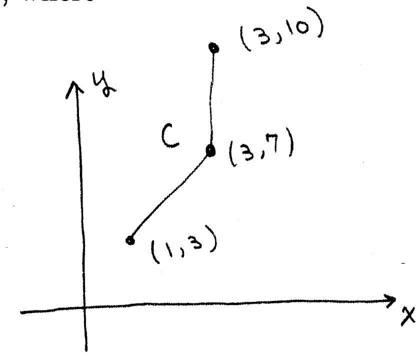
$$= 60 \int_0^1 x^2 (x - x^2)^2 \, dx = 60 \int_0^1 (x^4 - 2x^5 + x^6) \, dx$$

$$= 60 \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7} \right) = \frac{60}{105}$$

$$= \boxed{\frac{4}{7}}$$

$$m = \frac{7-3}{3-1} = \frac{4}{2} = 2$$

7. (10 points) Let C be the curve made up of two line segments: the first from $(1,3)$ to $(3,7)$, and the second from $(3,7)$ to $(3,10)$. Evaluate $\int_C \vec{F}(x,y)\cdot d\vec{r}$, where $\vec{F}(x,y) = (x^2y+2)\hat{\imath} + (1-xy)\hat{\jmath}$.



$$\vec{F}\cdot d\vec{r} = (x^2y+2)\,dx + (1-xy)\,dy$$

C_1: $y-3 = 2(x-1)$ or $y = 2x+1$, $1 \le x \le 3$

C_2: $x = 3$, $7 \le y \le 10$

$$\int_C \vec{F}\cdot d\vec{r} = \int_1^3 \left[x^2(2x+1)+2\right]dx + \left[1-x(2x+1)\right]2\,dx$$

$$+ \int_7^{10} (9y+2)(0)\,dy + (1-3y)\,dy$$

$$= \int_1^3 \left(2x^3+x^2+2+2-4x^2-2x\right)dx + \int_7^{10}(1-3y)\,dy$$

$$= \int_1^3 \left(2x^3-3x^2-2x+4\right)dx + \int_7^{10}(1-3y)\,dy$$

$$= \left(\tfrac{1}{2}x^4 - x^3 - x^2 + 4x\right)\Big|_1^3 + \left(y - \tfrac{3}{2}y^2\right)\Big|_7^{10}$$

$$= \left(\frac{33}{2} - \frac{5}{2}\right) + \left(-140 + \frac{133}{2}\right) = \boxed{\frac{-119}{2} = -59.5}$$