Math 233 - Test 2

March 12, 2026

Name Key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Find the function $\vec{r}(t)$ that satifies the following conditions:

$$\vec{r}\,'(t) = te^{-t^2}\,\hat{\imath} + \cos(2t)\,\hat{\jmath} + \frac{1}{1+t^2}\,\hat{k}; \quad \vec{r}(0) = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}.$$

$\int te^{-t^2}\,dt = -\frac{1}{2}e^{-t^2} + C_1$

$u = -t^2$

$du = -2t\,dt$

$\int \cos(2t)\,dt = \frac{1}{2}\sin(2t) + C_2$

$\int \frac{1}{1+t^2}\,dt = \tan^{-1}t + C_3$

$\vec{r}(t) = \left(-\frac{1}{2}e^{-t^2} + C_1\right)\hat{\imath} + \left(\frac{1}{2}\sin 2t + C_2\right)\hat{\jmath} + \left(\tan^{-1}t + C_3\right)\hat{k}$

$\vec{r}(0) = \langle 3, -2, 4\rangle \Rightarrow C_1 = \frac{7}{2},\ C_2 = -2,\ C_3 = 4$

$\vec{r}(t) = \left(\frac{7}{2} - \frac{1}{2}e^{-t^2}\right)\hat{\imath} + \left(\frac{1}{2}\sin 2t - 2\right)\hat{\jmath} + \left(\tan^{-1}t + 4\right)\hat{k}$

2. (8 points) The graph of the $\vec{r}(t) = (t + \sin t)\,\hat{\imath} + (1 - \cos t)\,\hat{\jmath}$ is an example of a curve called a *cycloid*. Find the arc length of the cycloid from $t = 0$ to $t = \pi$. (Your integral should be easy if you use the trig identity $1 + \cos t = 2\cos^2(t/2)$.)

$\vec{r}\,'(t) = (1 + \cos t)\hat{\imath} + \sin t\,\hat{\jmath}$

$\|\vec{r}\,'(t)\| = \sqrt{(1+\cos t)^2 + \sin^2 t} = \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 + 2\cos t}$

$= \sqrt{4\cos^2\left(\frac{t}{2}\right)}$

$= 2\left|\cos\frac{t}{2}\right|$

Arc Length $= \int_0^{\pi} 2\left|\cos\frac{t}{2}\right|\,dt = 2\int_0^{\pi}\cos\frac{t}{2}\,dt$

$= 4\sin\frac{t}{2}\Big|_0^{\pi} = 4 - 0 = \boxed{4}$

3. (10 points) Let $\vec{r}(t) = 2t\,\hat{\imath} + \cos(3t)\,\hat{\jmath} - \sin(3t)\hat{k}$. Starting from $t = 0$, find the arc-length parameter $s(t)$, and then reparameterize $\vec{r}$ in terms of s.

$$\vec{r}\,'(t) = 2\hat{i} - 3\sin 3t\,\hat{j} - 3\cos 3t\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{4 + 9\sin^2 3t + 9\cos^2 3t} = \sqrt{4+9} = \sqrt{13}$$

$$s(t) = \int_0^t \sqrt{13}\,du = \sqrt{13}\,t$$

$$\boxed{s(t) = \sqrt{13}\,t, \quad t \geq 0}$$

$$t = \frac{s}{\sqrt{13}} \Rightarrow \boxed{\vec{R}(s) = \frac{2s}{\sqrt{13}}\hat{i} + \cos\left(\frac{3s}{\sqrt{13}}\right)\hat{j} - \sin\left(\frac{3s}{\sqrt{13}}\right)\hat{k}}$$

Follow-up question: What is the magnitude of the derivative of your reparameterized function?

WHEN A FUNCTION IS PARAMETERIZED IN TERMS OF ARC LENGTH, THE MAG. OF ITS DERIVATIVE IS ALWAYS $\boxed{1}$

4. (8 points) Let $\vec{r}(t) = (t^3 - 3t)\,\hat{\imath} + (3t - 5)^2\,\hat{\jmath} - 2t\hat{k}$. Compute $\hat{T}(2)$ and confirm that it is a unit vector.

$$\vec{r}\,'(t) = (3t^2 - 3)\hat{i} + 2(3t-5)(3)\hat{j} - 2\hat{k}$$

$$\vec{r}\,'(2) = 9\hat{i} + 6\hat{j} - 2\hat{k} \qquad \|\vec{r}\,'(2)\| = \sqrt{81+36+4} = \sqrt{121} = 11$$

$$\hat{T}(2) = \frac{\vec{r}\,'(2)}{\|\vec{r}\,'(2)\|} = \boxed{\frac{9}{11}\hat{i} + \frac{6}{11}\hat{j} - \frac{2}{11}\hat{k}}$$

$$\|\hat{T}(2)\| = \sqrt{\frac{81}{121} + \frac{36}{121} + \frac{4}{121}} = \sqrt{\frac{121}{121}} = 1 \checkmark$$

5. (8 points) A projectile is launched from 6 feet above the ground at an angle of 12° with the horizontal. The projectile is to have a range of 250 feet. Find the initial velocity. **To get credit for this problem, you must set up and use the vector-valued function that gives the position of the projectile at time t.** (Use $g = 32\,\text{ft/sec}^2$.)

$$\vec{r}(t) = V_0 \cos 12^\circ t \ \hat{i} + \left(-16t^2 + V_0 \sin 12^\circ t + 6\right)\hat{j}$$

$$V_0 \cos 12^\circ t = 250 \longrightarrow V_0 = \frac{250}{t \cos 12^\circ}$$

$$-16t^2 + V_0 \sin 12^\circ t + 6 = 0$$

$$-16t^2 + 250 \tan 12^\circ + 6 = 0$$

$$t^2 = \frac{250 \tan 12^\circ + 6}{16} \approx 3.696$$

$$t \approx 1.9225 \text{ sec}$$

$$V_0 = \frac{250}{t \cos 12^\circ}$$

$$\Rightarrow \boxed{V_0 \approx 132.9 \text{ FT/SEC}}$$

6. (8 points) For $n \geq 2$, think about the graph of $y = x^n$. Compute the curvature at $x = 1$. Then show that $\lim\limits_{n\to\infty} \kappa(1) = 0$.

$$\frac{dy}{dx} = nx^{n-1}$$

$$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

$$K(x) = \frac{n(n-1)|x|^{n-2}}{\left[1 + (nx^{n-1})^2\right]^{3/2}}$$

$$\boxed{K(1) = \frac{n(n-1)}{(1+n^2)^{3/2}}}$$

$$\lim_{n\to\infty} \frac{n(n-1)}{(1+n^2)^{3/2}} \quad \infty/\infty \quad \text{MORE WORK.}$$

$$= \lim_{n\to\infty} \frac{n(n-1)}{(1+n^2)^{3/2}} \cdot \frac{\frac{1}{(n^2)^{3/2}}}{\frac{1}{(n^2)^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n^2}}{\left(\frac{1}{n^2} + 1\right)^{3/2}} = \frac{0-0}{1}$$

$$= \boxed{0}$$

$a_N = 0$

7. (4 points) An object is moving along a line and its acceleration vector is given by

$$\vec{a}(t) = \frac{e^t\,\hat{\imath} + te^t\,\hat{\jmath}}{\sqrt{1+t^2}}. \quad = a_T \hat{T}(t)$$

Assume that $\vec{a}(t)$ has the same direction as $\hat{T}(t)$ and find the tangential component of the acceleration, a_T.

$\|\vec{a}(t)\| = |a_T| = a_T$

$$\boxed{a_T = e^t}$$

$$\|\vec{a}(t)\| = \frac{1}{\sqrt{1+t^2}}\sqrt{e^{2t} + t^2 e^{2t}} = \frac{e^t\sqrt{1+t^2}}{\sqrt{1+t^2}} = e^t$$

8. (6 points) Imagine you are moving at a constant (nonzero) speed.

   (a) Explain how it is possible that you may be accelerating. $a_T = 0$

   YOU ARE CHANGING DIRECTION : $a_N \neq 0$.

   (b) Assuming that you are accelerating, describe the relationship between your velocity and acceleration vectors. THEY ARE ORTHOGONAL!

   YOUR VELOCITY VECTOR HAS THE DIRECTION OF $\hat{T}$, BUT WHEN $a_T = 0$, YOUR ACCELERATION VECTOR HAS THE DIRECTION OF $\hat{N}$.

9. (10 points) Compute $\hat{N}(0)$ when $\vec{r}(t) = \sqrt{2}t\,\hat{\imath} + e^t\,\hat{\jmath} + e^{-t}\,\hat{k}$.

$$\vec{r}\,'(t) = \sqrt{2}\,\hat{\imath} + e^t\,\hat{\jmath} - e^{-t}\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

$$\hat{T}(t) = \frac{\sqrt{2}\,\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}}{e^t + e^{-t}}$$

$$\hat{T}'(t) = \frac{(e^t + e^{-t})(e^t\hat{\jmath} + e^{-t}\hat{k}) - (\sqrt{2}\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k})(e^t - e^{-t})}{(e^t + e^{-t})^2}$$

$$\hat{T}'(0) = \frac{2(\hat{\jmath} + \hat{k})}{4} = \frac{1}{2}\hat{\jmath} + \frac{1}{2}\hat{k}.$$

$$\|\hat{T}'(0)\| = \sqrt{\tfrac{1}{2}}$$

$$\boxed{\hat{N}(0) = \frac{\hat{T}'(0)}{\|\hat{T}'(0)\|} = \frac{1}{\sqrt{2}}\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}}$$

10. (10 points) Each of these equations defines a surface in 3-space. Briefly describe each surface.

(a) $10 + 3x^2 - 8y^2 + 5z^2 = 0$

Fix x : Hyperbolas Fix z : Hyperbolas

Fix y : $3x^2 + 5z^2 = 8y^2 - 10$ Ellipses for big enough y

Two-sheet Hyperboloid

(b) $4x^2 - \frac{1}{4}y^2 - \frac{1}{9}z^2 = -1$

Fix x : $4x^2 + 1 = \frac{1}{4}y^2 + \frac{1}{9}z^2$ Ellipses at any x.

Fix y : Hyperbolas Fix z : Hyperbolas

One-sheet Hyperboloid

(c) $x = (z+1)^2 + 1$

Parabolic cylinder with generating curve in xz-plane and rulings parallel to y-axis.

(d) $4x^2 + 25y^2 - z = 1$

$z = 4x^2 + 25y^2 - 1$

Elliptical paraboloid opening up z-axis with vertex at $(0,0,-1)$

(e) $y = 4x^2 - z^2$

Fix x : Parabolas

Fix y : Hyperbolas

Fix z : Parabolas

Hyperbolic Paraboloid

11. (6 points) Let $f(x,y) = \dfrac{\sqrt{x^2+y^2-9}}{x}$.

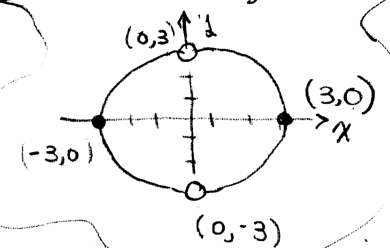
(a) What is the domain of f?

$\{(x,y) : x^2+y^2 \geq 9 \text{ and } x \neq 0\}$

(b) Sketch the level curve $\underbrace{f(x,y)}$ $= 0$.

$x^2+y^2-9=0 \Rightarrow x^2+y^2=9$

Circle centered at $(0,0)$ with radius 3, $x \neq 0$.



(c) Describe the level curve $f(x,y) = 2$.

$\sqrt{x^2+y^2-9} = 2x$

$x^2+y^2-9 = 4x^2$

$-3x^2+y^2=9$ ← Hyperbola, but must have $x^2+y^2 \geq 9$ and $x \neq 0$

item (6 points) Let $g(x, y, z) = \dfrac{x - y + z}{2x + y - z}$.

(a) Evaluate $g(1, 0, -2)$.

$$g(1,0,-2) = \frac{1-0-2}{2+0+2} = \boxed{-\frac{1}{4}}$$

(b) What is the domain of g?

$2x+y-z \neq 0$.

$$\boxed{\text{DOMAIN} = \{(x,y,z) : z \neq 2x+y\}}$$

(c) Describe the level surface $g(x, y, z) = 2$.

$$\frac{x-y+z}{2x+y-z} = 2 \Rightarrow x-y+z = 4x+2y-2z \Rightarrow$$

$3x+3y-3z=0$
PLANE W/
$\vec{n} = \hat{i}+\hat{j}-\hat{k}$, EXCEPT
WE MUST HAVE
$2x+y-z \neq 0$

12. (10 points) Find each limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x - y - 1}{\sqrt{x - y} - 1}$

% MORE WORK.

$$\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1} \cdot \frac{\sqrt{x-y}+1}{\sqrt{x-y}+1}$$

$$= \lim_{(x,y)\to(2,1)} \frac{\cancel{(x-y-1)}(\sqrt{x-y}+1)}{\cancel{x-y-1}} = \sqrt{2-1}+1 = \boxed{2}$$

(b) $\displaystyle\lim_{(x,y)\to(3,3)} \frac{x^4 - y^4}{x^2 - y^2}$

% MORE WORK

$$= \lim_{(x,y)\to(3,3)} \frac{(x^2+y^2)\cancel{(x^2-y^2)}}{\cancel{x^2-y^2}} = \lim_{(x,y)\to(3,3)} (x^2+y^2)$$

$$= 9+9 = \boxed{18}$$