Math 233 - Test 3

April 16, 2026

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Consider the limit $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$.

   (a) Compute the limit along a line of the form $y = kx$, where $k \neq 0$.

   $$\lim_{x\to 0} \frac{x^2+k^2x^2}{kx^2} = \lim_{x\to 0} \frac{1+k^2}{k} = \boxed{\frac{1+k^2}{k}}$$

   (b) Determine the limit along the parabola $y = x^2$.

   $$\lim_{x\to 0} \frac{x^2+x^4}{x^3} = \underbrace{\lim_{x\to 0} \frac{1+x^2}{x}}_{\frac{1}{0} \Rightarrow \text{ Some kind of inf limits.}}$$

   $$\lim_{x\to 0^-} \frac{1+x^2}{x} = -\infty \qquad \lim_{x\to 0^+} \frac{1+x^2}{x} = +\infty$$

   $\boxed{\text{Limit DNE}}$

   (c) Does the limit exist? Say why or why not.

   Limit DNE. Limit depends on path. See (a) & (b).

2. (8 points) Find the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)}{x^4+(y-1)^2}$$

0/0 Let's try some paths.

Along $y=1$: $\lim_{x\to 0} \frac{0}{x^4} = \lim_{x\to 0} 0 = 0$

Along $y=x^2+1$: $\lim_{x\to 0} \frac{x^4}{x^4+x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$

Two different limits along two paths $\Rightarrow$ $\boxed{\text{Limit DNE.}}$

3. (4 points) Determine all points at which f is continuous.

$$f(x,y) = \frac{\ln(xy) + \sin(xy)}{2x^2 - y^2}$$

WE MUST HAVE $xy > 0$ FOR $\ln(xy)$ TO EXIST (AND BE CONT.).

WE MUST HAVE $2x^2 - y^2 \neq 0$ TO AVOID DIVISION BY ZERO.

$\boxed{f \text{ IS CONTINUOUS EVERYWHERE WHERE } xy > 0 \text{ AND } 2x^2 \neq y^2.}$

4. (7 points) Compute both first-order partial derivatives at the point $(1,2)$. Write your final answers in exact form (not decimal approximations).

$$g(x,y) = x \ln \sqrt{2x^2 + 3y^2}$$

$\ln\sqrt{2x^2+3y^2} = \frac{1}{2}\ln(2x^2+3y^2)$

$g(x,y) = \frac{1}{2} x \ln(2x^2+3y^2)$

$g_x(x,y) = \frac{1}{2}\ln(2x^2+3y^2) + \frac{1}{2}x\left(\frac{4x}{2x^2+3y^2}\right) \Rightarrow \boxed{g_x(1,2) = \frac{\ln(14)}{2} + \frac{2}{14}}$

$g_y(x,y) = \left(\frac{1}{2}x\right)\left(\frac{6y}{2x^2+3y^2}\right) \Rightarrow \boxed{g_y(1,2) = \frac{6}{14}}$

5. (8 points) Let $z = e^{xy}$.

(a) In order to compute $\frac{\partial^2 z}{\partial y \partial x}$, which 1st-order partial derivative should be computed first and why?

X FIRST BECAUSE $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$.

OR YOU COULD SAY IT DOESN'T MATTER BECAUSE BOTH MIXED PARTIALS WILL BE CONTINUOUS AND EQUAL.

(b) Find $\frac{\partial^2 z}{\partial y \partial x}$.

$\frac{\partial z}{\partial x} = ye^{xy}$

$\frac{\partial}{\partial y}(ye^{xy}) = \boxed{e^{xy} + xye^{xy}}$

6. (3 points) Briefly (but thoroughly) describe an interpretation of an x-partial derivative.

THE X-PARTIAL DERIVATIVE AT A POINT IS THE SLOPE OF THE GRAPH AT THAT POINT IN THE DIRECTION OF $\hat{L}$.

7. (4 points)

(a) There exist functions for which the mixed partial derivatives are not equal. Suppose $f_{xy}(a,b) \neq f_{yx}(a,b)$. What must also be true about f_{xy} and f_{yx}?

$\rightarrow$ f_{xy} OR f_{yx} MUST NOT BE CONTINUOUS IN A NEIGHBORHOOD OF (a,b).

(b) For a polynomial $P(x,y)$, would you expect $P_{xyx} = P_{yxy}$? Say why or why not.

ABSOLUTELY NOT. THEY WOULD EACH BE CONTINUOUS, BUT DERIVATIVES ARE NOT MIXED THE SAME WAY.

8. (8 points) The distance from the point $(2,-3)$ to the point (x,y) is given by

$$z = D(x,y) = \sqrt{(x-2)^2 + (y+3)^2}.$$

(a) Find the total differential dz.

$$dz = D_x(x,y)\,dx + D_y(x,y)\,dy$$

$$\boxed{dz = \frac{x-2}{\sqrt{(x-2)^2+(y+3)^2}}\,dx + \frac{y+3}{\sqrt{(x-2)^2+(y-3)^2}}\,dy}$$

(b) Use differentials to approximate Δz when (x,y) changes from $(6,0)$ to $(5.95, 0.15)$

$$\Delta z \approx \frac{x-2}{\sqrt{(x-2)^2+(y+3)^2}}\,\Delta x + \frac{y+3}{\sqrt{(x-2)^2+(y+3)^2}}\,\Delta y$$

$x = 6$ $\quad \Delta x = -0.05$

$y = 0$ $\quad \Delta y = 0.15$

$$\Longrightarrow \Delta z \approx \frac{4}{5}(-0.05) + \frac{3}{5}(0.15) = \frac{0.25}{5} = \boxed{\frac{1}{20}}$$

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9. (6 points) Suppose that z is a differentiable function of x and y. Further suppose that x and y are differentiable functions of the independent variables s, t, and u. Write the chain rule formulas for the derivatives of z with respect to each independent variable.

① $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$, ② $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$,

③ $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

10. (6 points) Suppose that z is implicitly defined as a function of x and y by the equation

$$xyz + 2yz^3 - 8xy = 7x - 2y + x^2y^3z.$$

Use partial derivatives to determine $\partial z/\partial y$.

$F(x,y,z) = xyz + 2yz^3 - 8xy - 7x + 2y - x^2y^3z$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \boxed{\frac{-(xz + 2z^3 - 8x + 2 - 3x^2y^2z)}{xy + 6yz^2 + x^2y^3}}$$

11. (8 points) Find the maximum value of the directional derivative of $f(x,y) = \dfrac{x+y}{y+1}$ at the point $(0,1)$.

$\|\vec{\nabla} f(0,1)\|$

$\vec{\nabla} f(x,y) = \frac{1}{y+1}\hat{i} + \frac{(y+1)-(x+y)}{(y+1)^2}\hat{j} = \frac{1}{y+1}\hat{i} + \frac{1-x}{(y+1)^2}\hat{j}$

$\vec{\nabla} f(0,1) = \frac{1}{2}\hat{i} + \frac{1}{4}\hat{j}$ $\quad \|\vec{\nabla} f(0,1)\| = \sqrt{\frac{1}{4} + \frac{1}{16}}$

$= \boxed{\frac{\sqrt{5}}{4}}$

12. (8 points) Find a set of parametric equations for the line normal to the graph of $\underbrace{x^2 - 8xyz + y^2 + 6z^2}_{F(x,y,z)} = 0$ at the point $P\,(1,1,1)$.

$F(x,y,z) \longrightarrow$ OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$.

$$\vec{\nabla} F(x,y,z) = (2x - 8yz)\hat{i} + (-8xz + 2y)\hat{j} + (-8xy + 12z)\hat{j}$$

$$n = \vec{\nabla} F(1,1,1) = -6\hat{i} - 6\hat{j} + 4\hat{k}$$

I'LL USE $\vec{n} = 3\hat{i} + 3\hat{j} - 2\hat{k}$

NORMAL LINE : $\boxed{\begin{aligned} x &= 3t + 1 \\ y &= 3t + 1 \\ z &= -2t + 1 \end{aligned}}$

13. (5 points) The surface of a smooth hill is modeled by the equation

$$h(x, y) = 5000 - 0.001x^2 - 0.004y^2.$$

A hiker is at the point $(500, 300, 4390)$. In which direction should the hiker proceed in order to descend at the greatest rate?

$$\vec{\nabla} h(x,y) = -0.002x\,\hat{i} - 0.008y\,\hat{j}$$

$$\vec{\nabla} h(500, 300) = -\hat{i} - 2.4\hat{j}$$

$\uparrow$ TO DESCEND MOST QUICKLY, go OPPOSITE THE GRADIENT VECTOR.

$$\boxed{\hat{i} + 2.4\hat{j}}$$

14. (5 points) Find the critical point(s) of $f(x, y) = \sqrt{x^2 + y^2}$. DOMAIN IS $\mathbb{R}^2$.

$$f_x(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$\vec{\nabla} f(x,y) = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$$

$\vec{\nabla} f(x,y)$ DNE WHEN $(x,y) = (0,0)$.

$\vec{\nabla} f(x,y) = \vec{0}$ NOWHERE.

$\boxed{\text{CRIT PT: } (0,0)}$

15. (8 points) Find and classify the critical point(s) of $g(x, y) = x^2 - xy - y^2 - 3x - y$. Determine all relative extreme values and saddle points.

$$g_x(x,y) = 2x - y - 3 = 0$$
$$g_y(x,y) = -x - 2y - 1 = 0$$

$\Rightarrow$

$$(2x - y = 3)\,2$$
$$+ \quad x + 2y = -1$$
$$5x = 5$$
$$x = 1$$
$$y = 2(1) - 3 = -1$$

$(1,-1)$ ONLY ONE CRIT PT.

$$D(x,y) = \begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = -5$$

$$D(1,-1) < 0$$

$\Rightarrow$ $(1,-1,-1)$ IS A SADDLE PT.

$$g(1,-1) = 1 + 1 - 1 - 3 + 1$$
$$= -1$$

The following problem is a take-home problem. It is due April 21, 2026. You must work on your own.

16. (6 points) Suppose we would like to convert the differentiable function $w = f(x, y)$ to polar coordinates by using $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Use the appropriate chain rule to find formulas for $\partial w/\partial r$ and $\partial w/\partial \theta$.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\, f_x + \sin\theta\, f_y$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\, f_x + r\cos\theta\, f_y$$

(b) Solve your equations in part (a) for f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial \theta$.

SEE BACK SIDE $\longrightarrow$

(c) Show that $(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$.

$$(f_x)^2 + (f_y)^2 = \sin^2\theta \left(\frac{\partial w}{\partial r}\right)^2 + \cancel{\frac{2}{r}\sin\theta\cos\theta \frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}} + \frac{1}{r^2}\cos^2\theta\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$+ \cos^2\theta\left(\frac{\partial w}{\partial r}\right)^2 - \cancel{\frac{2}{r}\sin\theta\cos\theta\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}} + \frac{1}{r^2}\cos^2\theta\left(\frac{\partial w}{\partial \theta}\right)^2$$

Now use $\sin^2\theta + \cos^2\theta = 1$ To get $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$.

(d) Let $f(x, y) = \tan^{-1}(y/x)$. Compute and simplify the expression in part (c) (in rectangular coordinates or polar coordinates, you decide).

$$f_x(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$f_y(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$(f_x)^2 + (f_y)^2 = \frac{x^2+y^2}{(x^2+y^2)^2}$$

$$\boxed{= \frac{1}{x^2+y^2} = \frac{1}{r^2}}$$

$$\frac{\partial w}{\partial r} = \cos\theta\, f_x + \sin\theta\, f_y$$

$$\frac{\partial w}{\partial \theta} = -r\sin\theta\, f_x + r\cos\theta\, f_y$$

FIRST MULTIPLY TOP EQUATION BY $r\sin\theta$, BOTTOM BY $\cos\theta$,

AND ADD THEM:

$$r\sin\theta\,\frac{\partial w}{\partial r} + \cos\theta\,\frac{\partial w}{\partial \theta} = r\, f_y$$

$$\boxed{f_y = \sin\theta\,\frac{\partial w}{\partial r} + \frac{1}{r}\cos\theta\,\frac{\partial w}{\partial \theta}}$$

NEXT MULTIPLY TO EQUATION BY $-r\cos\theta$, BOTTOM BY $\sin\theta$,

AND ADD THEM:

$$-r\cos\theta\,\frac{\partial w}{\partial r} + \sin\theta\,\frac{\partial w}{\partial \theta} = -r\, f_x$$

$$\boxed{f_x = \cos\theta\,\frac{\partial w}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial w}{\partial \theta}}$$