Math 233 - Test 3

Name _____

April 16, 2026

Score _____

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Consider the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{xy}$.

   (a) Compute the limit along a line of the form $y = kx$, where $k \neq 0$.

   (b) Determine the limit along the parabola $y = x^2$.

   (c) Does the limit exist? Say why or why not.

2. (8 points) Find the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)}{x^4+(y-1)^2}$$

3. (4 points) Determine all points at which f is continuous.

$$f(x,y) = \frac{\ln(xy) + \sin(xy)}{2x^2 - y^2}$$

4. (7 points) Compute both first-order partial derivatives at the point $(1,2)$. Write your final answers in exact form (not decimal approximations).

$$g(x,y) = x \ln \sqrt{2x^2 + 3y^2}$$

5. (8 points) Let $z = e^{xy}$.

   (a) In order to compute $\dfrac{\partial^2 z}{\partial y \partial x}$, which 1st-order partial derivative should be computed first and why?

   (b) Find $\dfrac{\partial^2 z}{\partial y \partial x}$.

6. (3 points) Briefly (but thoroughly) describe an interpretation of an x-partial derivative.

7. (4 points)

   (a) There exist functions for which the mixed partial derivatives are not equal. Suppose $f_{xy}(a,b) \neq f_{yx}(a,b)$. What must also be true about f_{xy} and f_{yx}?

   (b) For a polynomial $P(x,y)$, would you expect $P_{xyx} = P_{yxy}$? Say why or why not.

8. (8 points) The distance from the point $(2,-3)$ to the point (x,y) is given by

$$z = D(x,y) = \sqrt{(x-2)^2 + (y+3)^2}.$$

   (a) Find the total differential dz.

   (b) Use differentials to approximate Δz when (x,y) changes from $(6,0)$ to $(5.95, 0.15)$

9. (6 points) Suppose that z is a differentiable function of x and y. Further suppose that x and y are differentiable functions of the independent variables s, t, and u. Write the chain rule formulas for the derivatives of z with respect to each independent variable.

10. (6 points) Suppose that z is implicitly defined as a function of x and y by the equation

$$xyz + 2yz^3 - 8xy = 7x - 2y + x^2y^3z.$$

Use partial derivatives to determine $\partial z/\partial y$.

11. (8 points) Find the maximum value of the directional derivative of $f(x, y) = \dfrac{x+y}{y+1}$ at the point $(0, 1)$.

12. (8 points) Find a set of parametric equations for the line normal to the graph of $x^2 - 8xyz + y^2 + 6z^2 = 0$ at the point $P\,(1, 1, 1)$.

13. (5 points) The surface of a smooth hill is modeled by the equation

$$h(x, y) = 5000 - 0.001x^2 - 0.004y^2.$$

A hiker is at the point $(500, 300, 4390)$. In which direction should the hiker proceed in order to descend at the greatest rate?

14. (5 points) Find the critical point(s) of $f(x, y) = \sqrt{x^2 + y^2}$.

15. (8 points) Find and classify the critical point(s) of $g(x, y) = x^2 - xy - y^2 - 3x - y$. Determine all relative extreme values and saddle points.

The following problem is a take-home problem. It is due April 21, 2026. You must work on your own.

16. (6 points) Suppose we would like to convert the differentiable function $w = f(x, y)$ to polar coordinates by using $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Use the appropriate chain rule to find formulas for $\partial w/\partial r$ and $\partial w/\partial\theta$.

(b) Solve your equations in part (a) for f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial\theta$.

(c) Show that $(f_x)^2 + (f_y)^2 = \left(\dfrac{\partial w}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial w}{\partial\theta}\right)^2$.

(d) Let $f(x, y) = \tan^{-1}(y/x)$. Compute and simplify the expression in part (c) (in rectangular coordinates or polar coordinates, you decide).