Math 233 - Final Exam A Name key

May 8, 2026 Score ______

Show all work to receive full credit. Supply explanations where necessary. This portion of the test is due May 14. You must work individually.

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1. (10 points) A stunt performer is preparing a new stunt in which she will be launched from a cannon through a large flaming ring. For her preliminary designs, she will ignore air resistance.

   She will be launched from 6 ft above the ground, and the center of the flaming ring is 60 ft downrange and 40 ft high. Find her initial speed and launch angle so that she passes through the center of the ring at the highest point of her path. **To get credit for this problem, you must set up and use the vector-valued function that gives the position of the projectile at time** t. (Use $g = 32\,\text{ft/sec}^2$.)

$$\vec{r}(t) = V_0 \cos\theta\, t\, \hat{i} + \left(-16t^2 + V_0 \sin\theta\, t + 6\right)\hat{j}$$

MUST SIMULTANEOUSLY HAVE

$$-32t + V_0 \sin\theta = 0$$

$$V_0 \cos\theta\, t = 60$$

$$-16t^2 + \underbrace{V_0 \sin\theta}_{32t}\, t + 6 = 40$$

$$-16t^2 + 32t^2 + 6 = 40$$

$$t^2 = \frac{34}{16}$$

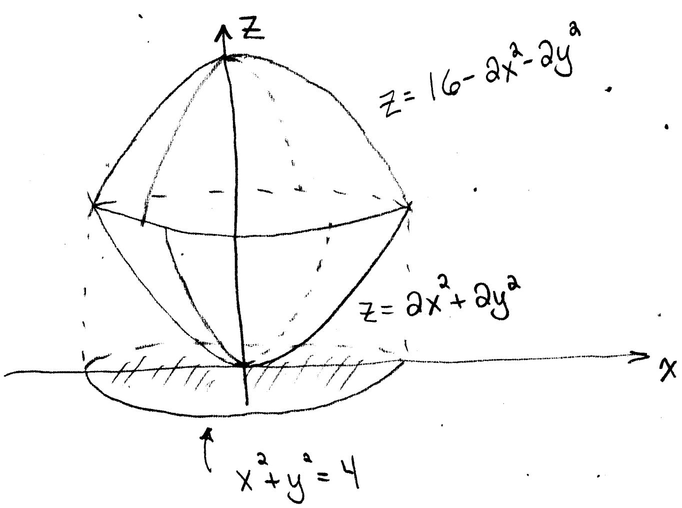
$$t = \frac{\sqrt{34}}{4}$$

$$\frac{V_0 \sin\theta}{V_0 \cos\theta} = \tan\theta = \frac{32t}{60/t} = \frac{32t^2}{60} = \frac{68}{60}$$

$$\tan\theta = \frac{68}{60} \Rightarrow \boxed{\theta \approx 48.58^\circ}$$

$$V_0 = \frac{60}{\cos\theta\, t} = \frac{4(60)}{\sqrt{34}\,\cos\theta} \approx \boxed{62.21 \text{ FT/s}}$$

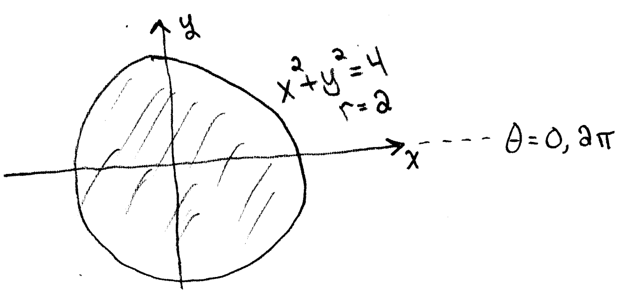
2. (10 points) A solid in space is bounded by the paraboloids $z = 16 - 2x^2 - 2y^2$ and $z = 2x^2 + 2y^2$. The density of the solid at the point (x, y, z) is given by $\rho(x, y, z) = \sqrt{x^2 + y^2}$. Find the mass of the solid. (Helpful advice: Use cylindrical coordinates.)



$$16 - 2x^2 - 2y^2 = 2x^2 + 2y^2$$

$$\Downarrow$$

$$x^2 + y^2 = 4$$



$$\iiint_R 1 \, dV$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=2r^2}^{z=16-2r^2} r^2 \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^2 (16 - 4r^2) \, dr$$

$$= 2\pi \int_0^2 (16r^2 - 4r^4) \, dr$$

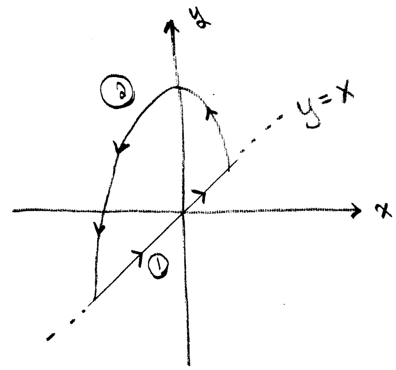
$$= 2\pi \left(\frac{16}{3} r^3 - \frac{4}{5} r^5 \right) \Bigg|_0^2$$

$$= 2\pi \left(\frac{128}{3} - \frac{128}{5} \right)$$

$$= \boxed{\frac{512\pi}{15} \text{ units}^3}$$

3. (10 points) Let $\vec{F}(x,y) = (2x+y)\,\hat{\imath} + (xy+4)\,\hat{\jmath}$ and let C be the positively-oriented boundary of the plane region enclosed by the graphs of $y = x$ and $y = 2 - x^2$. Consider the line integral

$$\int_C \vec{F}(x,y)\cdot d\vec{r}.$$



(a) Evaluate the line integral without using Green's theorem.

① $\vec{F}(x,y)\cdot d\vec{r} = (2x+y)\,dx + (xy+4)\,dy$

1ST PART... $y=x$, $dy=dx$

$$\int_{-2}^{1}(2x+x)\,dx + (x^2+4)\,dx = \int_{-2}^{1}(3x+x^2+4)\,dx$$

$$= \frac{3}{2}x^2 + \frac{1}{3}x^3 + 4x \Big|_{-2}^{1} = \frac{35}{6} + \frac{14}{3} = \frac{21}{2}$$

$x = 2-x^2$

$x^2+x-2=0$

$(x-1)(x+2)=0$

$x=1,\ x=-2$

② 2ND PART... $y=2-x^2$, $dy=-2x\,dx$

$$\int_{1}^{-2}(2x+2-x^2)\,dx + \left[x(2-x^2)+4\right](-2x\,dx) = \int_{1}^{-2}\left(-6x+2-5x^2+2x^4\right)dx$$

$$= -3x^2+2x-\frac{5}{3}x^3+\frac{2}{5}x^5\Big|_{1}^{-2} = \frac{-232}{15}+\frac{34}{15} = \frac{-198}{15}$$

LINE INTEGRAL $= \dfrac{21}{2} - \dfrac{198}{15} = \dfrac{-27}{10}$

(b) Use Green's theorem to evaluate the line integral.

$$\iint_R (y-1)\,dA = \int_{x=-2}^{x=1}\int_{y=x}^{y=2-x^2}(y-1)\,dy\,dx$$

$$= \int_{-2}^{1} \frac{1}{2}y^2 - y\,\Big|_{x}^{2-x^2}\,dx$$

$$= \int_{-2}^{1}\left[\frac{1}{2}(2-x^2)^2-(2-x^2)-\frac{1}{2}x^2+x\right]dx$$

$$= \int_{-2}^{1}\left(2-2x^2+\frac{1}{2}x^4-2+x^2-\frac{1}{2}x^2+x\right)dx$$

$$= \int_{-2}^{1}\left(-\frac{3}{2}x^2+x+\frac{1}{2}x^4\right)dx = -\frac{1}{2}x^3+\frac{1}{2}x^2+\frac{1}{10}x^5\Big|_{-2}^{1} = \left(\frac{1}{10}-\frac{14}{5}\right) = \boxed{-\frac{27}{10}}$$