

**Math 233 - Final Exam B**  
May 14, 2026

Name key  
Score \_\_\_\_\_

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Let  $\vec{r}(t) = \frac{1}{3}t^3 \hat{i} + 2t^2 \hat{j}$  for  $t \geq 0$ .

(a) Find the arc-length parameter,  $s(t)$ , starting from  $t = 0$ .

$$\vec{r}'(t) = t^2 \hat{i} + 4t \hat{j}$$

$$\begin{aligned} \|\vec{r}'(t)\| &= \sqrt{t^4 + 16t^2} \\ &= t\sqrt{t^2 + 16}, \quad t \geq 0 \end{aligned}$$

$$s(t) = \int_0^t u\sqrt{u^2 + 16} \, du$$

$$w = u^2 + 16$$

$$dw = 2u \, du$$

$$\frac{1}{2} dw = u \, du$$

$$s(t) = \frac{1}{2} \int_{16}^{t^2+16} \sqrt{w} \, dw$$

$$\begin{aligned} s(t) &= \frac{1}{3} u^{3/2} \Big|_{16}^{t^2+16} \\ &= \frac{1}{3} (t^2+16)^{3/2} - \frac{64}{3} \end{aligned}$$

$$s(t) = \frac{1}{3} (t^2+16)^{3/2} - \frac{64}{3}$$

(b) Use your arc length parameter to find the length of the curve from  $t = 0$  to  $t = 3$ .

$$s(3) = \frac{1}{3} (25)^{3/2} - \frac{64}{3}$$

$$= \frac{125}{3} - \frac{64}{3} = \frac{61}{3} \approx 20.3$$

2. (10 points) Find and classify the critical points of  $g(x,y) = x^2 + 2y^2 - x^2y$ . Identify all relative extreme values and saddle points.

$$g_x(x,y) = 2x - 2xy = 0 \Rightarrow 2x(1-y) = 0 \quad x = 0 \text{ or } y = 1$$

$$g_y(x,y) = 4y - x^2 = 0$$

$$\begin{array}{l} \downarrow \\ 4y = 0 \\ y = 0 \end{array} \quad \begin{array}{l} 4 - x^2 = 0 \\ x = \pm 2 \end{array}$$

$$D(x,y) = \begin{vmatrix} 2-2y & -2x \\ -2x & 4 \end{vmatrix} = 8 - 8y - 4x^2$$

$$(0,0), (2,1), (-2,1)$$

(0,0)

$$D(0,0) = 8 > 0, \quad g_{xx}(0,0) = 2 > 0$$

$g(0,0) = 0$  IS A REL MIN.

( $\pm 2, 1$ )

$$D(\pm 2, 1) = -16 < 0$$

$(\pm 2, 1, 2)$  ARE SADDLE POINTS

3. (10 points) Consider the surface described by the equation  $z = 2e^{4x^2+2xy-4y}$ .

(a) Find an equation of the plane tangent to the surface at the point  $(1, 2, 2)$ .

$$F(x, y, z) = z - 2e^{4x^2+2xy-4y}. \quad \text{Our surface is the level surface}$$

$$F(x, y, z) = 0.$$

$\vec{\nabla} F(x, y, z)$  is normal!

$$\vec{\nabla} F(x, y, z) = -2(8x + 2y)e^{4x^2+2xy-4y} \hat{i} - 2(2x-4)e^{4x^2+2xy-4y} \hat{j} + \hat{k}$$

$$\vec{\nabla} F(1, 2, 2) = -2(8+4)e^{\hat{i}} - 2(2-4)e^{\hat{j}} + \hat{k} = -24\hat{i} + 4\hat{j} + \hat{k}$$

$$\vec{n} = -24\hat{i} + 4\hat{j} + \hat{k} \quad \text{You use } 24\hat{i} - 4\hat{j} - \hat{k} \dots$$

$$24x - 4y - z = 14$$

(b) Find a set of parametric equations for the line normal to the surface at the point  $(1, 2, 2)$ .

$$x = 24t + 1$$

$$y = -4t + 2$$

$$z = -t + 2$$

4. (10 points) Use Lagrange multipliers to find the minimum and maximum values of  $f(x, y) = xy$  on the circle  $\underbrace{x^2 + y^2 - 8 = 0}_{g(x, y)}$ .

$$\vec{\nabla} f(x, y) = y\hat{i} + x\hat{j}$$

$$\vec{\nabla} g(x, y) = 2x\hat{i} + 2y\hat{j}$$

$$y = \lambda 2x \Rightarrow 2\lambda = \frac{y}{x} = \frac{x}{y} \Rightarrow x^2 = y^2$$

$$x = \lambda 2y$$

$$x^2 + y^2 = 8$$

$$2x^2 = 8$$

$$x = \pm 2 \Rightarrow y^2 = 4,$$

$$y = \pm 2, \mp 2$$

Four crit. pts:

$$(2, 2), (2, -2), (-2, 2), (-2, -2)$$

$$f(2, 2) = f(-2, -2) = 4 \leftarrow \text{MAX}$$

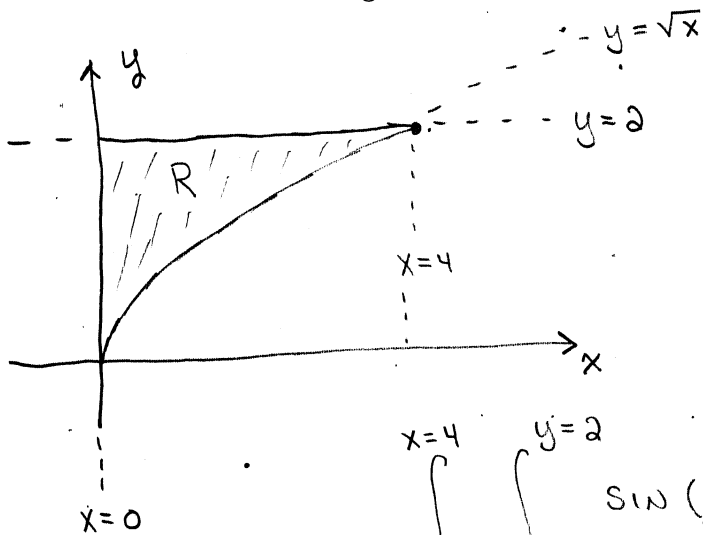
$$f(2, -2) = f(-2, 2) = -4 \leftarrow \text{MIN}$$

5. (10 points) Consider the double integral

$$\iint_R \sin(y^3) dA,$$

where  $R$  is the 1st quadrant region enclosed by the graphs of  $x = 0$ ,  $y = 2$ , and  $y = \sqrt{x}$ .

Sketch the region of integration, write the double integral as an iterated integral with both orders of integration, and evaluate either one of your iterated integrals.



$$2 = \sqrt{x} \Rightarrow x = 4$$

$$\int_{x=0}^{x=4} \int_{y=\sqrt{x}}^{y=2} \sin(y^3) dy dx$$

$$= \int_{y=0}^{y=2} \int_{x=0}^{x=y^2} \sin(y^3) dx dy$$

$$= \int_0^2 y^2 \sin(y^3) dy$$

$$u = y^3$$

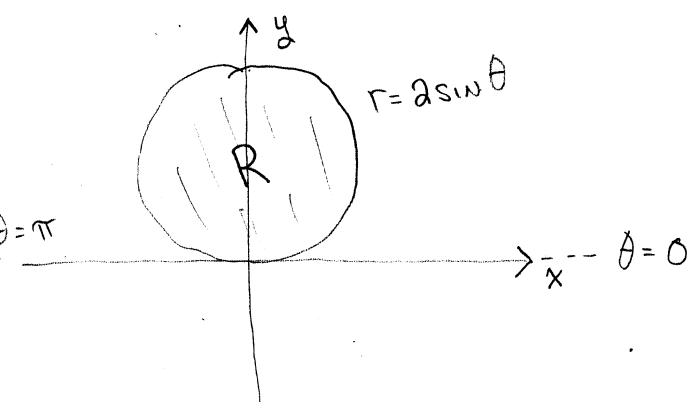
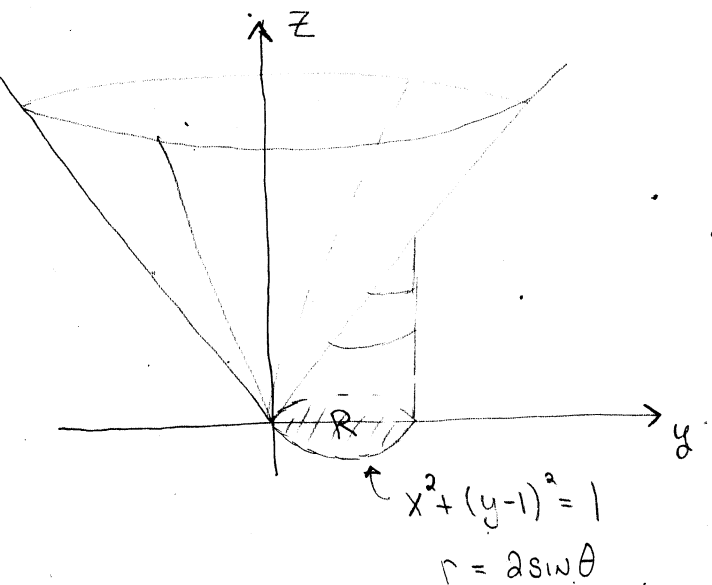
$$du = 3y^2 dy$$

$$\frac{1}{3} du = y^2 dy$$

$$\frac{1}{3} \int_0^8 \sin u du = \frac{1}{3} \cos u \Big|_8^0$$

$$= \frac{1}{3} - \frac{1}{3} \cos 8 \approx 0.382$$

6. (10 points) Use a double integral in polar coordinates to find the volume of the solid bounded above by the cone  $z = \sqrt{x^2 + y^2}$ , bounded below by the  $xy$ -plane, and lying inside the cylinder  $x^2 + y^2 = 2y$ . (Some helpful information is given below.)



$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y-1)^2 = 1$$

$$\text{Volume} = \iint_R \sqrt{x^2 + y^2} \, dA$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} r^2 \, dr \, d\theta$$

$$= \int_0^{\pi} \left. \frac{1}{3} r^3 \right|_{r=0}^{r=2\sin\theta} d\theta$$

$$= \frac{8}{3} \int_0^{\pi} \sin^3 \theta \, d\theta$$

$$= \left. \frac{8}{9} \cos^3 \theta - \frac{8}{3} \cos \theta \right|_0^{\pi}$$

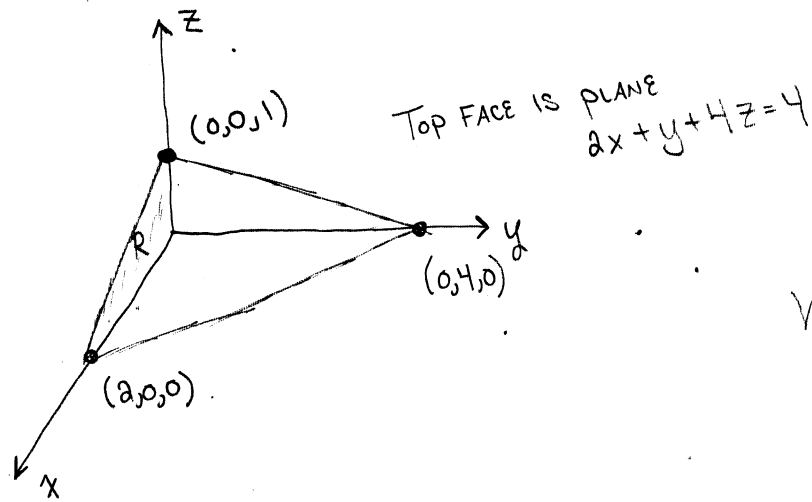
$$= -\frac{8}{9} + \frac{8}{3} - \frac{8}{9} + \frac{8}{3} = \frac{16}{3} - \frac{16}{9}$$

$$= \boxed{\frac{32}{9}}$$

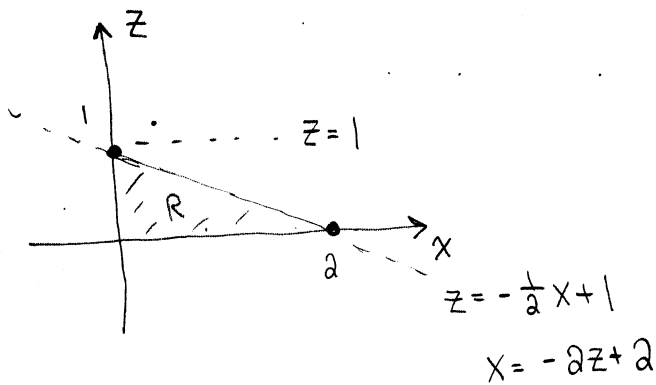
These ideas may be useful...

- Complete the square to find the center and radius of the circle  $x^2 + y^2 = 2y$ .
- In polar coordinates, the circle  $x^2 + y^2 = 2y$  has equation  $r = 2 \sin \theta$ .
- $\int \sin^3 \theta \, d\theta = \frac{1}{3} \cos^3 \theta - \cos \theta + C$

7. (10 points) Use a triple integral to find the volume of the tetrahedron bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ , and  $2x + y + 4z = 4$ .



$$\text{Volume} = \int_{z=0}^{z=1} \int_{x=0}^{x=2-2z} \int_{y=0}^{y=4-2x-4z} dy \, dx \, dz$$



$$= \int_{z=0}^{z=1} \int_{x=0}^{x=2-2z} (4-2x-4z) \, dx \, dz$$

$$= \int_0^1 \left( 4x - x^2 - 4xz \right) \Big|_{x=0}^{x=2-2z} dz$$

$$= \int_0^1 (2-2z)(4-2+2z-4z) \, dz$$

$$= \int_0^1 (2-2z)^2 \, dz = \int_0^1 (4-8z+4z^2) \, dz$$

$$= 4 - 4 + \frac{4}{3} = \boxed{\frac{4}{3}}$$