MTH 236 Assignment 10 key

1) $y_1 v_1' + y_2 v_2' = 0$

$y_1' v_1' + y_2' v_2' = g$

Cramer's rule...

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-g(x)\,y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}$$

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{g(x)\,y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}$$

2) "Is similar to" is an equivalence relation.

① $A \sim A$: $A = IAI^{-1}$ (Reflexive)

② Suppose $A \sim B$. Then $A = PBP^{-1}$. It follows that $B = P^{-1}AP = QAQ^{-1}$ where $Q = P^{-1}$, and $B \sim A$.

(Symmetric)

③ Suppose $A \sim B$ and $B \sim C$. Then $A = PBP^{-1}$ and $B = QCQ^{-1}$. It follows that

(Transitive)

$$A = P(QCQ^{-1})P^{-1} = (PQ)C(PQ)^{-1} = MCM^{-1} \text{ where } M = PQ. \therefore A \sim C.$$

3) SUPPOSE $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ IS DIAGONALIZABLE BY

$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, WHERE P IS NONSINGULAR.

THEN

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

OR

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a\delta_1 & b\delta_2 \\ c\delta_1 & d\delta_2 \end{pmatrix}$$

CASE 1: $\delta_1 \neq 0$, $\delta_2 \neq 0$.

THEN $c = d = 0$ AND THIS CONTRADICTS P IS NONSINGULAR.

CASE 2: $\delta_1 \neq 0$, $\delta_2 = 0$

THEN $c = 0$ AND $\therefore$ $a = 0$. THIS CONTRADICTS P IS NONSING.

CASE 3: $\delta_1 = 0$, $\delta_2 \neq 0$

THEN $d = 0$ AND $\therefore$ $b = 0$. THIS CONTRADICTS P IS NONSING.

CASE 4: $\delta_1 = \delta_2 = 0$.

THEN $c = 0$ AND $d = 0$. THIS CONTRADICTS P IS NONSING.

THERE ARE NO POSSIBILITIES LEFT. WE CANNOT HAVE

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a\delta_1 & b\delta_2 \\ c\delta_1 & d\delta_2 \end{pmatrix}$$ WITH NONSING P.

4) $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix}$$

$$= (4-\lambda)\left[(1-\lambda)(8-\lambda)+6\right] + 2(8-\lambda) - 12 + 6\left[-2-2(1-\lambda)\right]$$

$$= (4-\lambda)(1-\lambda)(8-\lambda) + 24 - 6\lambda + 16 - 2\lambda - 12 - 12 - 12 + 12\lambda$$

$$= 32 - 36\lambda + 4\lambda^2 - 8\lambda + 9\lambda^2 - \lambda^3 + 4 + 4\lambda$$

$$= 36 - 40\lambda + 13\lambda^2 - \lambda^3$$

$$\boxed{\text{CHAR POLY} = p(\lambda) = 36 - 40\lambda + 13\lambda^2 - \lambda^3}$$

$p(2) = 36 - 80 + 52 - 8 = 0$

$\Rightarrow \lambda = 2$ IS AN EIGENVALUE.

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

CORRESPONDING EIGENVECTORS ARE $\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$ AND $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$$\boxed{\text{EIGENSPACE HAS BASIS } \left\langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle}$$

5) Char poly of A is

$$p(\lambda) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$$\boxed{p(\lambda) = (5-\lambda)^2(3-\lambda)(1-\lambda)}$$

6) $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has only $\lambda = 1$ (mult 2)

7) Suppose A is a square matrix for which $A^2 = O$.

Let λ be any eigenvalue of A

Then $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} = \vec{0}$.

It follows that $A(A\vec{x}) = A^2\vec{x} = A\lambda\vec{x}$

$$= \lambda A\vec{x} = \lambda^2\vec{x} = 0$$

Since $\vec{x} \neq \vec{0}$, we must have $\lambda = 0$.

8) $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-5-\lambda)(1-\lambda)+9\right] - 3\left[-3(1-\lambda)+9\right] + 3\left[-9+3(5+\lambda)\right]$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) - 3(6+3\lambda) + 3(6+3\lambda)$$

$$= (1-\lambda)(\lambda+2)^2 = 0 \Rightarrow \lambda = 1,\ \lambda = -2 \text{ (mult 2)}$$

$\lambda = 1 \ldots$

$$A - I = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = -2 \ldots$

$$A + 2I = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{X}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},\quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

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$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$ From Sage.

$$PDP^{-1} = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} = A \quad \checkmark$$