Math 236 - Assignment 11

April 24, 2024

Name ____________________

Score ________________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due May 1.

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1. Let $D = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_m)$ be an $m \times m$ diagonal matrix. Use induction to prove that $D^n = \operatorname{diag}(\delta_1^n, \delta_2^n, \ldots, \delta_m^n)$.

2. Use the definition of the matrix exponential and the result of problem 1 to prove that
$$e^D = \operatorname{diag}(e^{\delta_1}, e^{\delta_2}, \ldots, e^{\delta_m}),$$
for any $m \times m$ diagonal matrix D.

3. Let $A = \begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix}$. Diagonalize A and then compute e^A.

4. Find the eigenvalues and eigenvectors of C. Feel free to use technology to find the characteristic polynomial and its zeros. Also feel free to use technology for any RREF.
$$C = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

5. Let V be the vector space of polynomials defined on $[a, b]$. (Notice that unless we put a cap on the degree of the polynomials, V is infinite dimensional, which is fine!) Define a "product" on V by
$$\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)\,dx.$$
Show that $\langle \cdot, \cdot \rangle$ is an inner product.

6. The set of "vectors,"
$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots\},$$
is mutually orthogonal with respect to the inner product
$$\langle f(x), g(x) \rangle = \int_0^{2\pi} f(x)g(x)\,dx.$$
Pick any two distinct elements and show they are orthogonal.

7. The first three Legendre polynomials are
$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_3(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$
Show that these polynomials are mutually orthogonal with respect to the inner product
$$\langle P_m(x), P_n(x) \rangle = \int_{-1}^1 P_m(x)P_n(x)\,dx.$$

8. Compute the norm of each of the Legendre polynomials above.