MTH 236 - Assignment 11 Key

1) $D = \begin{pmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_m \end{pmatrix}$. Use induction to prove

$$D^n = \begin{pmatrix} \delta_1^n & & & 0 \\ & \delta_2^n & & \\ & & \ddots & \\ 0 & & & \delta_m^n \end{pmatrix}.$$

① $D^1 = \begin{pmatrix} \delta_1^1 & & & 0 \\ & \delta_2^1 & & \\ & & \ddots & \\ 0 & & & \delta_m^1 \end{pmatrix} = D$ ✓

② Assume $D^k = \begin{pmatrix} \delta_1^k & & & 0 \\ & \delta_2^k & & \\ & & \ddots & \\ 0 & & & \delta_m^k \end{pmatrix}$.

③ $D^{k+1} = DD^k = \begin{pmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_m \end{pmatrix}\begin{pmatrix} \delta_1^k & & & 0 \\ & \delta_2^k & & \\ & & \ddots & \\ 0 & & & \delta_m^k \end{pmatrix}$

$$= \begin{pmatrix} \delta_1^{k+1} & 0 & 0 & \cdots & 0 \\ 0 & \delta_2^{k+1} & 0 & \cdots & \\ \vdots & & \ddots & & \\ \vdots & & & & \delta_m^{k+1} \end{pmatrix}$$

By carrying out the multiplication.

2) D IS AN $m \times m$ DIAGONAL MATRIX.

$$e^D = I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots$$

$$= \begin{pmatrix} 1 + \delta_1 + \frac{1}{2!}\delta_1^2 + \frac{1}{3!}\delta_1^3 + \cdots & & & 0 \\ & 1 + \delta_2 + \frac{1}{2!}\delta_2^2 + \frac{1}{3!}\delta_2^3 + \cdots & & \\ & & \ddots & \\ 0 & & & 1 + \delta_m + \frac{1}{2!}\delta_m^2 + \frac{1}{3!}\delta_m^3 + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\delta_1} & & & 0 \\ & e^{\delta_2} & & \\ & & \ddots & \\ 0 & & & e^{\delta_m} \end{pmatrix}$$

3) $A = \begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{pmatrix}$

EXPANDING DOWN 2ND COLUMN,

$$\det(A - \lambda I) = (-2-\lambda)\left[(7-\lambda)(-8-\lambda) + 54\right]$$

$$= (-2-\lambda)\left[\lambda^2 + \lambda - 2\right] = -(\lambda+2)^2(\lambda-1) = 0$$

$\lambda_1 = -2$, MULT 2; $\lambda_2 = 1$ MULT 1

$\lambda = -2 \ldots$

$$A - \lambda I = \begin{pmatrix} 9 & 0 & -3 \\ -9 & 0 & 3 \\ 18 & 0 & -6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{X}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{X}_2 = \begin{pmatrix} 1/3 \\ 0 \\ 1 \end{pmatrix} \text{ OR } \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

CONTINUED...

PROBLEM 3 CONTINUED...

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} 6 & 0 & -3 \\ -9 & -3 & 3 \\ 18 & 0 & -9 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{X}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 3 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 3 & 1 & -1 \\ -2 & 0 & 1 \\ 3 & 0 & -1 \end{pmatrix} \quad \text{(FROM SAGE)}$$

$$A = PDP^{-1}$$

$$\Rightarrow e^A = P \begin{pmatrix} e^{-2} & 0 & 0 \\ 0 & e^{-2} & 0 \\ 0 & 0 & e \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 3e^1 - 2e^{-2} & 0 & -e^1 + e^{-2} \\ -3e^1 + 3e^{-2} & e^{-2} & e - e^{-2} \\ 6e^1 - 6e^{-2} & 0 & -2e^1 + 3e^{-2} \end{pmatrix}$$

4) $$C = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

USING SAGE FOR ALL BELOW.

$$\det(C - \lambda I) = \lambda^4 - 16\lambda^3 + 76\lambda^2 - 176\lambda + 160$$

$$\det(C - \lambda I) = 0 \Rightarrow \lambda = 10,\ \lambda = 2,\ \lambda = 2 - 2i,\ \lambda = 2 + 2i$$

$\lambda = 10 \ldots$

$$\text{RREF}(C - \lambda I) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 2 \ldots$

$$\text{RREF}(C - \lambda I) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 2 \pm 2i \ldots$

$$\text{RREF}(C - \lambda I) = \begin{pmatrix} 1 & 0 & 0 & \mp i \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \pm i \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{X}_3 = \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix} \ \& \ \vec{X}_4 = \begin{pmatrix} -i \\ -1 \\ i \\ 1 \end{pmatrix}$$

CORRESPONDS TO $\lambda = 2 + 2i$

CORRESPONDS TO $\lambda = 2 - 2i$

5) V IS THE VECTOR SPACE OF POLYNOMIALS ON $[a,b]$.

DEFINE $\langle f(x), g(x) \rangle = \int_a^b f(x) g(x)\, dx$.

SHOW THAT $\langle \cdot, \cdot \rangle$ IS AN INNER PRODUCT.

(i) $$\langle g(x), f(x) \rangle = \underbrace{\int_a^b g(x) f(x)\, dx = \int_a^b f(x) g(x)\, dx}_{\text{MULT OF REAL \#'S IS COMMUTATIVE.}} = \langle f(x), g(x) \rangle$$

(ii) $\langle \alpha f_1(x) + \beta f_2(x), g(x) \rangle$

$$= \int_a^b (\alpha f_1(x) + \beta f_2(x)) g(x)\, dx = \cdots =$$

$$= \alpha \int_a^b f_1(x) g(x)\, dx + \beta \int_a^b f_2(x) g(x)\, dx$$

$$= \alpha \langle f_1(x), g(x) \rangle + \beta \langle f_2(x), g(x) \rangle$$

(iii) $\langle f(x), f(x) \rangle = \int_a^b (f(x))^2\, dx \geq 0$ BECAUSE $(f(x))^2 \geq 0$.

CONTINUED ...

PROBLEM 5 CONTINUED ...

(iv) IF $f(x) = 0$ ON $[a,b]$, THEN

$$\int_a^b [f(x)]^2\,dx = \int_a^b 0\,dx = 0.$$

IF $\int_a^b [f(x)]^2\,dx = 0$, THEN $f(x) = 0$ ON $[a,b]$.

IF NOT, SINCE $[f(x)]^2$ IS CONTINUOUS AND > 0 SOMEWHERE, THERE WOULD BE AN INTERVAL $[\alpha,\beta] \subseteq [a,b]$ ON WHICH $[f(x)]^2 > 0$ AND WE WOULD HAVE

$$\int_a^b [f(x)]^2\,dx \geq \int_\alpha^\beta [f(x)]^2\,dx > 0.$$

6) I'll choose $f(x) = \cos 2x$ and $g(x) = \sin 5x$.

$$\langle f(x), g(x) \rangle = \underbrace{\int_0^{2\pi} \cos 2x \sin 5x \, dx = \int_0^{2\pi}}_{\text{PRODUCT-TO-SUM FORMULA}} \frac{1}{2}\left[\sin 7x + \sin 3x\right] dx$$

$$= \int_0^{2\pi} \frac{1}{2}\sin 7x \, dx + \int_0^{2\pi} \frac{1}{2}\sin 3x \, dx$$

$$= \frac{1}{14}\cos 7x \Big|_{2\pi}^{0} + \frac{1}{6}\cos 3x \Big|_{2\pi}^{0}$$

$$= \frac{1}{14} - \frac{1}{14} + \frac{1}{6} - \frac{1}{6} = 0 \checkmark$$

$$\langle \cos 2x, \sin 5x \rangle = 0$$

7)

i. $\langle P_0(x), P_1(x) \rangle = \int_{-1}^{1} x \, dx = \frac{1}{2}x^2 \Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$ ✓

ii. $\langle P_0(x), P_2(x) \rangle = \int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) dx = \frac{1}{2}x^3 - \frac{1}{2}x \Big|_{-1}^{1}$

$= \left(\frac{1}{2} - \frac{1}{2}\right) - \left(-\frac{1}{2} + \frac{1}{2}\right) = 0$ ✓

iii. $\langle P_1(x), P_2(x) \rangle = \int_{-1}^{1} \left(\frac{3}{2}x^3 - \frac{1}{2}x\right) dx = \frac{3}{8}x^4 - \frac{1}{4}x^2 \Big|_{-1}^{1}$

$= \left(\frac{3}{8} - \frac{1}{4}\right) - \left(\frac{3}{8} - \frac{1}{4}\right) = 0$ ✓

8) $\sqrt{\langle P_0(x), P_0(x) \rangle} = \left[\int_{-1}^{1} 1 \, dx\right]^{1/2} = \boxed{\sqrt{2}}$

$\sqrt{\langle P_1(x), P_1(x) \rangle} = \left[\int_{-1}^{1} x^2 \, dx\right]^{1/2} = \boxed{\sqrt{\frac{2}{3}}}$

$\sqrt{\langle P_2(x), P_2(x) \rangle} = \left[\int_{-1}^{1} \left(\frac{9}{4}x^4 - \frac{3}{2}x^2 + \frac{1}{4}\right) dx\right]^{1/2} = \left[2\int_{0}^{1} \left(\frac{9}{4}x^4 - \frac{3}{2}x^2 + \frac{1}{4}\right) dx\right]^{1/2}$

$= \left[2\left(\frac{9}{20} - \frac{3}{6} + \frac{1}{4}\right)\right]^{1/2} = \boxed{\sqrt{\frac{2}{5}}}$