Math 236 - Assignment 1

January 17, 2024

Name Key ______________

Show all work to receive full credit. Supply explanations when necessary. Do all computations by hand. This assignment is due January 24.

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1. Find the solution set by reducing to echelon form.

$$\begin{array}{rcrcrcl} x_1 & - & 3x_2 & + & 4x_3 & = & -4 \\ 3x_1 & - & 7x_2 & + & 7x_3 & = & -8 \\ -4x_1 & + & 6x_2 & - & x_3 & = & 7 \end{array}$$

   Solution

   The sequence of row operations

$$R_2^{(1)} = -3R_1^{(0)} + R_2^{(0)},$$

$$R_3^{(1)} = 4R_1^{(0)} + R_3^{(0)},$$

$$R_3^{(2)} = 3R_2^{(1)} + R_3^{(1)}$$

   reduces the original system to

$$\begin{array}{rcrcrcl} x_1 & - & 3x_2 & + & 4x_3 & = & -4 \\ & & 2x_2 & - & 5x_3 & = & 4 \\ & & & & 0 & = & 3. \end{array}$$

   This system is inconsistent. There is no solution.

2. Find the solution set by reducing to echelon form.

$$\begin{array}{rcrcrcl} x_1 & & & - & 3x_3 & = & 8 \\ 2x_1 & + & 2x_2 & + & 9x_3 & = & 7 \\ & & x_2 & + & 5x_3 & = & -2 \end{array}$$

   Solution

   The sequence of row operations

$$R_2^{(1)} = R_3^{(0)}, \quad R_3^{(1)} = R_2^{(0)},$$

$$R_3^{(2)} = -2R_1^{(0)} + R_3^{(1)},$$

$$R_3^{(3)} = -2R_2^{(1)} + R_3^{(2)},$$

   reduces the original system to

$$\begin{array}{rcrcrcl} x_1 & & & - & 3x_3 & = & 8 \\ & & x_2 & + & 5x_3 & = & -2 \\ & & & & 5x_3 & = & -5. \end{array}$$

   Backsolving gives the unique solution $(x_1, x_2, x_3) = (5, 3, -1)$.

3. Prove that each elementary row operation in Gauss's method is reversible.

   Solution

   If two rows are swapped, say rows i and j (that is, $R_i \longleftrightarrow R_j$), then they can be swapped back (that is, $R_j \longleftrightarrow R_i$).

   If a row is multiplied by a nonzero number k, then you can get back to the original row by multiplying by the nonzero constant $1/k$.

   If the new row R_j is obtained by muliplying R_i by a nonzero k and adding it to the old row R_j, then the old row R_j can be retrieved from the new R_j by adding $-kR_i$.

4. For which values of b are there no solutions, infinitely many solutions, or a unique solution?

$$\begin{array}{rcrcl} 2x & + & y & = & 7 \\ 8x & + & 4y & = & b \end{array}$$

   Solution

   Multiplying row 1 by -4 and adding it to row 2 results in the echelon form

$$\begin{array}{rcrcl} 2x & + & y & = & 7 \\ & & 0 & = & b-28 \end{array}$$

   For the system to be consistent, we must have $b = 28$. This would result in there being infinitely many solutions. If $b \neq 28$, the system is inconsistent and there are no solutions. It is not possible for there to be a single solution.

5. Consider the system shown below with variables x and y. Use geometric reasoning to explain why there are three possibilities: no solution, infinitely many solutions, unique solution.

$$\begin{array}{rcrcl} ax & + & by & = & c \\ dx & + & ey & = & f \end{array}$$

   Solution

   Assuming at least one of a, b is nonzero and at least one of d, e is nonzero, then each equation describes a line in the plane. There are three ways those lines could be related. (1) If they are distinct parallel lines, then they have no points in common, and the system has no solution. (2) If the two equations describe the exact same line, then every point on one line is also a point on the other, and the system has infinitely many solutions. (3) Finally, if the lines are distinct, non-parallel lines, then they must intersect, and the system has a single solution.

6. Find the solution set by reducing to echelon form. Write the solution set in vector notation, identifying a particular solution and the solution of the corresponding homogeneous system.

$$\begin{array}{rcrcrcrcl} x & & & - & z & & & = & 1 \\ & & y & + & 2z & - & w & = & 3 \\ x & + & 2y & + & 3z & - & w & = & 7 \end{array}$$

Solution

The sequence of row operations

$$R_3^{(1)} = -R_1^{(0)} + R_3^{(0)},$$

$$R_3^{(2)} = -2R_2^{(0)} + R_3^{(1)}$$

reduces the original system to

$$\begin{array}{rcrcrcl} x & & - & z & & & = 1 \\ & y & + & 2z & - & w & = 3 \\ & & & & & w & = 0. \end{array}$$

In this echelon form, z is a free variable. Backsolve using z as a parameter to get:

$$w = 0, \quad z = z, \quad y = 3 - 2z, \quad x = 1 + z.$$

In vector form, the solution can be written

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1+z \\ 3-2z \\ z \\ 0 \end{pmatrix}.$$

A particular solution is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix},$$

and the corresponding homogeneous equation has solution

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z, \quad z \in \mathbb{R}.$$

7. Find the coefficients a, b, and c so that the graph of $p(x) = ax^2 + bx + c$ passes through the points $(1, -6)$, $(2, -9)$, and $(-1, -12)$.

Solution

$$p(1) = -6 \quad \Longrightarrow \quad a + b + c = -6$$

$$p(2) = -9 \implies 4a + 2b + c = -9$$
$$p(-1) = -12 \implies a - b + c = -12$$

The linear system

$$\begin{array}{rcrcrcr} a & + & b & + & c & = & -6 \\ 4a & + & 2b & + & c & = & -9 \\ a & - & b & + & c & = & -12 \end{array}$$

has the unique solution $a = -2$, $b = 3$, $c = -7$.

8. Describe all functions $f(x) = ax^2 + bx + c$ such that $f(1) = 2$. Do so by writing $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \text{particular} + \text{homogeneous}$.

Solution

$f(1) = 2 \implies a + b + c = 2.$

This "system" is in echelon form with free variables b and c. Its solution is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 - b - c \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} b + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} c; \quad b, c \in \mathbb{R}.$$