Math 236 - Assignment 2

Name KEY ______________

January 24, 2024

Show all work to receive full credit. Supply explanations when necessary. Do all computations by hand unless otherwise indicated. This assignment is due January 31.

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1. For any real numbers x and y, we will say $x \sim y$ if and only if $x - y$ is an integer. Prove that $\sim$ is an equivalence relation.

   Solution

   Let x, y, and z be real numbers.

   Reflexive: $x \sim x$ because $x - x = 0$ and 0 is an integer.

   Symmetric: Assume $x \sim y$. Then $x - y = k$, where k is an integer. It follows that $y - x = -k$. Since $-k$ is an integer, we have $y \sim x$.

   Transitive: Assume $x \sim y$ and $y \sim z$. Then $x - y = k$ and $y - z = n$, where k and n are integers. It follows that $x - z = k + n$. Since $k + n$ is an integer, $x \sim z$.

2. Prove that a linear combination of three linear combinations of x, y, and z is a linear combination of x, y, and z.

   Solution

   Let α, β, γ, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, and c_3 be constants.

   A linear combination of three linear combinations has the form

   $$\alpha(a_1x + a_2y + a_3z) + \beta(b_1x + b_2y + b_3z) + \gamma(c_1x + c_2y + c_3z).$$

   Distribute and rearrange terms to get

   $$(\alpha a_1 + \beta b_1 + \gamma c_1)x + (\alpha a_2 + \beta b_2 + \gamma c_2)y + (\alpha a_3 + \beta b_3 + \gamma c_3)z,$$

   which is a linear combination of x, y, and z.

3. Find (by hand) the reduced row echelon form (RREF).

   $$\begin{pmatrix} 1 & 0 & 3 & 1 & 2 \\ 1 & 4 & 2 & 1 & 5 \\ 3 & 4 & 8 & 1 & 2 \end{pmatrix}$$

   Solution

   Here are Sage steps that reduce the matrix to RREF:

```
A=matrix(QQ,[[1,0,3,1,2],[1,4,2,1,5],[3,4,8,1,2]])
A.add_multiple_of_row(1,0,-1)
A.add_multiple_of_row(2,0,-3)
A.rescale_row(1,1/4)
A.add_multiple_of_row(2,1,-4)
A.rescale_row(2,-1/2)
A.add_multiple_of_row(0,2,-1)
```

The RREF is

$$\begin{pmatrix} 1 & 0 & 3 & 0 & -3/2 \\ 0 & 1 & -1/4 & 0 & 3/4 \\ 0 & 0 & 0 & 1 & 7/2 \end{pmatrix}.$$

4. Write the system as an augmented matrix. Then compute the RREF and determine the solution set. You may use technology to find the RREF.

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & - & x_5 & = & 1 \\ 3x_1 & - & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \end{array}$$

<u>Solution</u>

The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 3 & 1 & -1 & 1 \\ 3 & -1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

```
A=matrix([[1,2,3,1,-1,1],[3,-1,1,1,1,3]])
A.rref()
```

The RREF is

$$\begin{pmatrix} 1 & 0 & 5/7 & 3/7 & 1/7 & 1 \\ 0 & 1 & 8/7 & 2/7 & -4/7 & 0 \end{pmatrix}.$$

The free variables are x_3, x_4, and x_5. The parameterized general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -5/7 \\ -8/7 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -3/7 \\ -2/7 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} -1/7 \\ 4/7 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_5; \quad x_3, x_4, x_5 \in \mathbb{R}.$$

5. Say why the following matrix is NOT in RREF. Then reduce it to RREF. Finally, once it's reduced, show that no row is a linear combination of the other rows.

$$\begin{pmatrix} 1 & 2 & 4 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Solution

The entry in row 1, column 3 should be zero. Zeroing that entry, $R_1 = -4R_2 + R_1$, will reduce the matrix to RREF:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & -18 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

First let's show that R_1 is not a linear combination of R_2 and R_3. The combination

$$R_1 = \alpha R_2 + \beta R_2$$

is impossible because the first entry of $\alpha R_2 + \beta R_3$ is $\alpha\,0 + \beta\,0 = 0 \neq 1$.

Now let's show that R_2 is not a linear combination of R_1 and R_3. The combination

$$R_2 = \alpha R_1 + \beta R_3$$

is impossible because the third entry of $\alpha R_1 + \beta R_3$ is $\alpha\,0 + \beta\,0 = 0 \neq 1$.

Finally let's show that R_3 is not a linear combination of R_1 and R_2. The combination

$$R_3 = \alpha R_1 + \beta R_2$$

is impossible because the fourth entry of $\alpha R_1 + \beta R_2$ is $\alpha\,0 + \beta\,0 = 0 \neq 1$.

6. Give two distinct echelon forms of the matrix below. Be sure to say (or show) which sequence of row operations gave each form.

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 6 & 4 & 1 & 2 \\ 1 & 5 & 1 & 5 \end{pmatrix}$$

Solution

The operations

```
A.add_multiple_of_row(1,0,-3)
A.add_multiple_of_row(2,0,-1/2)
A.add_multiple_of_row(2,1,-9/2)
A.rescale_row(2,2)
```

result in the echelon form

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 19 & 70 \end{pmatrix}.$$

On the other hand, the operations

```
A.swap_rows(0,2)
A.add_multiple_of_row(1,0,-6)
A.add_multiple_of_row(2,0,-2)
A.add_multiple_of_row(2,1,-9/26)
A.rescale_row(2,26)
```

result in the echelon form
$$\begin{pmatrix} 1 & 5 & 1 & 5 \\ 0 & -26 & -5 & -28 \\ 0 & 0 & 19 & 70 \end{pmatrix}.$$

7. Reduce to RREF. Then express the third row of the RREF as a linear combination of the rows of the original matrix.
$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 0 \\ 3 & -1 & 0 \end{pmatrix}$$

Solution

In order to do this, we'll keep careful track of the row operations that are used to get the final third row. First, let's do $R_3^{(1)} = -3R_1^{(0)} + R_3^{(0)}$ and $R_2^{(1)} = \frac{1}{4}R_2^{(0)}$ in order to obtain
$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -7 & -3 \end{pmatrix}.$$
Next, we do $R_3^{(2)} = 7R_2^{(1)} + R_3^{(1)}$ leaving
$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$
Finally, we do $R_3^{(3)} = -\frac{1}{3}R_3^{(2)}$ to obtain
$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Row 3 is done! How is it related to the original rows?
$$R_3^{(3)} = -\frac{1}{3}\left(7R_2^{(1)} + R_3^{(1)}\right) = -\frac{1}{3}\left(\frac{7}{4}R_2^{(0)} - 3R_1^{(0)} + R_3^{(0)}\right).$$
Therefore,
$$\text{Final Row 3} = R_3^{(3)} = R_1^{(0)} - \frac{7}{12}R_2^{(0)} - \frac{1}{3}R_3^{(0)}.$$

In order to finish this problem, we do two more obvious steps to obtain the RREF
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. Three truck drivers (in perhaps 1960) went into a roadside cafe. One truck driver purchased four sandwiches, a cup of coffee, and ten doughnuts for \$8.45. Another driver purchased three sandwiches, a cup of coffee, and seven doughnuts for \$6.30. What did the third truck driver pay for a sandwich, a cup of coffee, and a doughnut? (If your approach to this problem involves matrices and elimination, you may use technology.)

Solution

Let s, c, and d represent the number of sandwiches, cups of coffee, and doughnuts, respectively. Then we have

$$\begin{array}{rcrcrcl} 4s & + & c & + & 10d & = & 8.45 \\ 3s & + & c & + & 7d & = & 6.30 \end{array}$$

and we must find the value of $s + c + d$.

Let's solve the linear system by using RREF:

$$\begin{pmatrix} 1 & 0 & 3 & 2.15 \\ 0 & 1 & -2 & -0.15 \end{pmatrix}.$$

Now adding row 1 to row 2 gives

$$s + c + d = 2.00.$$