Math 236 - Assignment 3

Name KEY ______________

January 31, 2024

Show all work to receive full credit. Supply explanations when necessary. Do all computations by hand unless otherwise indicated. This assignment is due February 7.

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1. Let V be the set of all vectors in $\mathbb{R}^3$ with the usual scalar multiplication. However, define addition '+' in V as follows:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 \\ z_1 \end{pmatrix}.$$

Show that V is NOT a vector space.

Solution

The vector addition is not commutative. Let

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

be arbitrary vectors in $\mathbb{R}^3$. Then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} a + p \\ b \\ c \end{pmatrix},$$

while

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} p + a \\ q \\ r \end{pmatrix}.$$

The first component of the sum is the same in both results, but the second and third components are not necessarily the same.

2. Show that the set of all 2×2 diagonal matrices

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

with the usual operations of matrix addition and scalar multiplication is a vector space.

Solution

Let's name the space V and verify that the 10 vector space properties hold in V.

Property 1: Take two arbitrary matrices in V and add them:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a + c & 0 \\ 0 & b + d \end{pmatrix}.$$

The result is a diagonal matrix in V.

Property 2: Refer to the addition shown above. Because real number addition is commutative, that result is equal to

$$\begin{pmatrix} c+a & 0 \\ 0 & d+b \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Property 3: Take three arbitrary matrices in V.

$$\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) + \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} (a+c)+e & 0 \\ 0 & (b+d)+f \end{pmatrix}.$$

In the final matrix, the addition of real numbers is associative. Therefore

$$\begin{pmatrix} (a+c)+e & 0 \\ 0 & (b+d)+f \end{pmatrix} = \begin{pmatrix} a+(c+e) & 0 \\ 0 & b+(d+f) \end{pmatrix} = \cdots = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \left(\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right).$$

Property 4: The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix in V and

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Therefore $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the "zero vector."

Property 5: For any given matrix in V, the diagonal matrix with opposite entries works as the additive inverse in V:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Property 6: Take an arbitrary diagonal matrix in V and multiply it by the scalar α:

$$\alpha \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix}.$$

The result is a diagonal matrix in V.

Property 7: Take an arbitrary diagonal matrix in V and multiply it by the sum of the scalars α and β:

$$(\alpha+\beta) \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)a & 0 \\ 0 & (\alpha+\beta)b \end{pmatrix}.$$

Now expand and rewrite:

$$\begin{pmatrix} \alpha a+\beta a & 0 \\ 0 & \alpha b+\beta b \end{pmatrix} = \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} + \begin{pmatrix} \beta a & 0 \\ 0 & \beta b \end{pmatrix} = \alpha \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \beta \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Property 8: Take two arbitrary matrices in V and the scalar α.

$$\alpha\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) = \alpha\begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} = \cdots = \alpha\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \alpha\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

Property 9: Take an arbitrary diagonal matrix in V and multiply it by the product of the scalars α and β:

$$(\alpha\,\beta)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} (\alpha\beta)a & 0 \\ 0 & (\alpha\beta)b \end{pmatrix}.$$

Now rewrite using the associative property of real number multiplication:

$$\begin{pmatrix} \alpha(\beta a) & 0 \\ 0 & \alpha(\beta b) \end{pmatrix} = \alpha\left(\beta\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right).$$

Property 10: Take an arbitrary matrix in V and multiply by the scalar 1:

$$1\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1a & 0 \\ 0 & 1b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

3. Show that the set of all differentiable functions (of a single variable) with the usual operations of function addition and multiplication by a real constant is a vector space.

   Solution

   The closure properties, 1 and 6, follow immediately from the calculus results that say

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

$$\frac{d}{dx}[\alpha\, f(x)] = \alpha\, f'(x).$$

   The remaining properties are inherited from the real number system because the functions under consideration are real-valued functions.

4. Show that the set $\mathbb{R}^+$ of positive real numbers is a vector space when we interpret the "sum", $x+y$, as the product of x and y, and we interpret scalar "multiplication", $k\cdot x$, as the kth power of x.

   Solution

   Let's name the vector space V and verify that the 10 vector space properties hold in V.

   Property 1: Take two positive real numbers x and y and "add" them: $x+y = xy$. Since the product of two positive real numbers is a positive real number, $x+y$ is in V.

Property 2: $x + y = y + x$ because multiplication of positive real numbers is commutative.

Property 3: $(x + y) + z = (xy)z = x(yz) = x + (y + z)$ because the multiplaction of positive real numbers is associative.

Property 4: The positive real number 1 is the zero vector in V: $x + 1 = x\,1 = 1x = x$.

Property 5: For any given positive real number, the positive real number $1/x$ works as the additive inverse in V: $(x + 1/x) = x\,(1/x) = 1$.

Property 6: Take an arbitrary positive real number in V and "multiply" it by the scalar α: $\alpha \cdot x = x^\alpha$. For any real number α, x^α is a positive real number. Therefore $\alpha \cdot x = x^\alpha$ is in V.

Property 7: Take an arbitrary positive real number in V and "multiply" it by the sum (the regular sum in $\mathbb{R}$) of the scalars α and β: $(\alpha+\beta)\cdot x = x^{\alpha+\beta} = x^\alpha x^\beta = (\alpha\cdot x)+(\beta\cdot x)$.

Property 8: Take two arbitrary positive real numbers in V and the scalar α: $\alpha\cdot(x+y) = (xy)^\alpha = x^\alpha y^\alpha = \alpha \cdot x + \alpha \cdot y$.

Property 9: Take an arbitrary positive real number in V and "multiply" it by the product (the regular product in $\mathbb{R}$) of the scalars α and β: $(\alpha\,\beta) \cdot x = x^{\alpha\beta} = x^{\beta\alpha} = (x^\beta)^\alpha = \alpha \cdot (\beta \cdot x)$.

Property 10: Take an arbitrary positive real number in V and "multiply" by the scalar 1: $1 \cdot x = x^1 = x$.

5. Each element in a vector space must have an additive inverse. Prove that for each element x in vector space V, its additive inverse is unique. Use only the ten vector space conditions! (Hint: Let y and z be the additive inverses of x, and then show that y must be equal to z.)

   Solution

   Suppose x has two additive inverses y and z. Then $x + y = 0$ and $x + z = 0$ and it follows that $x + y = x + z$. Now add y to both sides to get the following:

$$(x + y) + y = (x + z) + y$$

$$0 + y = x + (z + y)$$

$$y = x + (y + z)$$

$$y = (x + y) + z$$

$$y = 0 + z$$

$$y = z$$

6. Is this a subspace of P_2: $\{ax^2 + bx + c : a = 1\}$?

Solution

No way! An arbitrary element of the space would have the form $x^2 + bx + c$. (A polynomial whose leading coefficient is 1 is called *monic*.) If we multiply by any scalar α for which $\alpha \neq 1$, we get $\alpha x^2 + \alpha bx + \alpha c$, which is not in the set. (It is not monic.)

7. Determine if $\begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$ is in the span of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$. What about $\begin{pmatrix} -5 & 0 \\ -5 & -12 \end{pmatrix}$?

Solution

First, we look for constants a and b so that

$$a \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}.$$

There are no such constants because any linear combination of the two matrices will have a zero in the $(1,2)$-position, not the required 1.

For the second question, the answer is yes. It is easy to verify that $a = 9$ and $b = -7$ do the trick:

$$9 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - 7 \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ -5 & -12 \end{pmatrix}.$$

8. Parameterize the subspace's description. Then express the subspace as a span of vectors in $M_{2\times 2}$.

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : 2a - c - d = 0 \text{ and } a + 3b = 0 \right\}$$

Solution

Use the given conditions to say $a = -3b$ and $c = 2a - d = -6b - d$ to rewrite the description as follows

$$\left\{ \begin{pmatrix} -3b & b \\ -6b - d & d \end{pmatrix} : b, d \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -3 & 1 \\ -6 & 0 \end{pmatrix} b + \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} d : b, d \in \mathbb{R} \right\}$$

$$= \text{span} \left(\left\{ \begin{pmatrix} -3 & 1 \\ -6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right\} \right).$$

9. Suppose that U and W are subspaces of the vector space V. Prove that $U \cap W$ is a subspace of V. (Recall that '$\cap$' stands for the intersection. Every element in $U \cap W$ is in both U and W.)

Solution

Let's show that $U \cap W$ is closed under linear combinations.

Let x and y be arbitary elements of $U \cap W$. Then $x \in U$, $x \in W$, $y \in U$, and $y \in W$. Since U is a subspace, $\alpha x + \beta y \in U$ for any scalars α and β. Similarly, since W is a subspace, $\alpha x + \beta y \in W$. So $\alpha x + \beta y$ is in U **and** in W. That is, $\alpha x + \beta y \in U \cap W$, and we've shown that $U \cap W$ is closed under linear combinations.