Math 236 - Assignment 4

KEY ____________________

February 14, 2024

Show all work to receive full credit. Supply explanations when necessary. This assignment is due February 21.

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1. Determine whether the set is a linearly dependent or independent subset of $\mathcal{P}_2$.

$$\{2 + x + 7x^2, 3 - x + 2x^2, 4 - 3x^2\}$$

Solution

Let c_1, c_2, and c_3 be constants with

$$c_1(2 + x + 7x^2) + c_2(3 - x + 2x^2) + c_3(4 - 3x^2) = 0.$$

This equation is equivalent to the system

$$2c_1 + 3c_2 + 4c_3 = 0,\ c_1 - c_2 = 0,\ 7c_1 + 2c_2 - 3c_3 = 0.$$

Reduce the associated augmented matrix to RREF:

$$\begin{pmatrix} 2 & 3 & 4 & 0 \\ 1 & -1 & 0 & 0 \\ 7 & 2 & -3 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, $c_1 = c_2 = c_3 = 0$, and the set is linearly independent.

2. Determine whether the set is a linearly dependent or independent subset of $M_{2\times 2}$.

$$\left\{ \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} \right\}$$

Solution

Let c_1 and c_2 be constants with

$$c_1 \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$5c_1 + c_2 = 0,\ 4c_1 = 0,\ c_1 - c_2 = 0,\ 2c_1 + 4c_2 = 0.$$

The only solution is $c_1 = c_2 = 0$. The "vectors" are linearly independent.

3. Suppose that the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent set. Prove that $\{\vec{u}, \vec{u} + \vec{v}, \vec{u} + \vec{v} + \vec{w}\}$ is also a linearly independent set.

Solution

Let's look at

$$d_1\vec{u} + d_2(\vec{u} + \vec{v}) + d_3(\vec{u} + \vec{v} + \vec{w}) = 0.$$

Distribute and rearrange to get

$$(d_1 + d_2 + d_3)\vec{u} + (d_2 + d_3)\vec{v} + d_3\vec{w} = 0.$$

It follows from the linear independence of $\vec{u}$, $\vec{v}$, and $\vec{w}$ that

$$d_1 + d_2 + d_3 = 0, \quad d_2 + d_3 = 0, \quad d_3 = 0.$$

The only solution is $d_1 = d_2 = d_3 = 0$.

4. Suppose that all proper subsets of $A = \{\vec{x}, \vec{y}, \vec{z}\}$ (except the empty set) are linearly independent. Must it be true that A itself is linearly independent?

   Solution

   Nope! Here is a simple counterexample:

$$\{\hat{\imath}, \hat{\jmath}, \hat{k}, \hat{\imath} + \hat{\jmath} + \hat{k}\}.$$

5. Is this a basis for $\mathcal{P}_2$?

$$\{x^2 - x + 1, 2x + 1, 2x - 1\}$$

   Solution

   Let's first check linear independence:

$$c_1(x^2 - x + 1) + c_2(2x + 1) + c_3(2x - 1) = 0 \Longrightarrow c_1 = 0,\ -c_1 + 2c_2 + 2c_3 = 0,\ c_1 + c_2 - c_3 = 0.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   Yes! The "vectors" are linearly indpendent.

   Does the set span $\mathcal{P}_2$? Well, yes. We actually answered that with the work above. The coefficient matrix above is nonsingular, so there is a unique solution for any right-hand side, not just the zero vector.

   Another way to prove this is to just notice that the set contains 3 linearly independent vectors in a 3-dimensional space. They must form a basis for the space.

6. Represent $x + x^3$ with respect to the given basis for $\mathcal{P}_3$.

$$B = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$$

   Solution

We find c_1, c_2, c_3, and c_4 so that

$$c_1(1) + c_2(1+x) + c_3(1+x+x^2) + c_4(1+x+x^2+x^3) = x + x^3.$$

This gives the system

$$\begin{array}{ccccccccc} c_1 & + & c_2 & + & c_3 & + & c_4 & = & 0 \\ & & c_2 & + & c_3 & + & c_4 & = & 1 \\ & & & & c_3 & + & c_4 & = & 0 \\ & & & & & & c_4 & = & 1 \end{array}.$$

The system is in echelon form. Backsolving gives

$$c_4 = 1,\ c_3 = -1,\ c_2 = 1,\ c_1 = -1,$$

so that

$$\text{Rep}_B(x + x^3) = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

7. Find a basis for the subspace below. Prove that it is a basis.

$$M = \{a + bx + cx^2 + dx^3 : 2a + b - c - 2d = 0\}$$

Solution

First use the condition to rewrite M:

$$\begin{aligned} M &= \{a + bx + (2a + b - 2d)x^2 + dx^3 : a, b, d \in \mathbb{R}\} \\ &= \{a(1 + 2x^2) + b(x + x^2) + d(-2x^2 + x^3) : a, b, d \in \mathbb{R}\}. \end{aligned}$$

Now let $B = \langle 1 + 2x^2, x + x^2, -2x^2 + x^3 \rangle$.

B spans M?
Yes. This is clear from the final way in which M is written. Polynomials in M are linear combinations of the elements of B.

B is linearly independent?
Yes. Suppose

$$c_1(1 + 2x^2) + c_2(x + x^2) + c_3(-2x^2 + x^3) = 0.$$

Then, by equating coefficients, we have

$$c1 = 0,\ c_2 = 0,\ 2c_1 + c_2 - 2c_3 = 0,\ c_3 = 0.$$

It follows that $c_1 = c_2 = c_3 = 0$.

8. Find a basis for the vector space of symmetric 2×2 matrices.

Solution

$$S = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

It should be pretty clear that a possible basis is

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

9. Find a basis for the subspace of polynomials $p \in \mathcal{P}_3$ with $p(1) = 0$ and $p(2) = 0$. Prove that it is a basis.

Solution

Let's call the subspace W. Then

$$W = \{a + bx + cx^2 + dx^3 : a + b + c + d = 0 \text{ and } a + 2b + 4c + 8d = 0\}.$$

The solution set for the conditions can be found by using RREF:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 8 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -2 & -6 & 0 \\ 0 & 1 & 3 & 7 & 0 \end{pmatrix}.$$

Therefore $a = 2c + 6d$ and $b = -3c - 7d$.

Now let's parameterize the subspace:

$$\begin{aligned} W &= \{(2c + 6d) + (-3c - 7d)x + cx^2 + dx^3 : c, d \in \mathbb{R}\} \\ &= \{c(2 - 3x + x^2) + d(6 - 7x + x^3) : c, d \in \mathbb{R}\} \\ &= \text{span}\{2 - 3x + x^2, 6 - 7x + x^3\}. \end{aligned}$$

Let $B = \langle 2 - 3x + x^2, 6 - 7x + x^3 \rangle$. It is clear from the parameterization that B spans W. To check linear independence, suppose

$$c_1(2 - 3x + x^2) + c_2(6 - 7x + x^3) = 0.$$

It is easy to see that $c_1 = c_2 = 0$.

10. Find a basis for, and the dimension of, the solution set of the following system.

$$\begin{array}{rcrcrcrcl} x_1 & - & 4x_2 & + & 3x_3 & - & x_4 & = & 0 \\ 2x_1 & - & 8x_2 & + & 6x^3 & - & 2x_4 & = & 0 \end{array}$$

Solution

The second equation is two times the first, so let's ignore it. With the remaining equation, we can backsolve, using x_2, x_3, and x_4 as free variables.

$$x_1 = 4x_2 - 3x_3 + x_4, \quad x_2 = x_2, \quad x_3 = x_3, \quad x_4 = x_4.$$

The solution space is

$$S = \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4 \right\}.$$

A basis for the solution space is

$$B = \left\langle \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

$\dim(S) = 3$.

11. What is the dimension of each vector space (or subspace) from problems 7–9?

Solution

Problem 7: $\dim(M) = 3$
Problem 8: $\dim(M) = 3$
Problem 9: $\dim(W) = 2$