Math 236 - Assignment 5

Name ____________________

February 21, 2024

Score __________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due February 28.

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1. Find a basis for the row space, a basis for the column space, and the rank of the matrix A.
$$A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix}$$

2. Consider the matrix $M = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}$.
   (a) Determine whether the row $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is in the row space of M.
   (b) Find a basis for the row space of M.
   (c) Find the representation for the row $\begin{pmatrix} -3 & 8 & 27 \end{pmatrix}$ in terms of your basis.

3. Find a basis for the span of the following subset of $\mathbb{R}^3$.
$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix} \right\}$$

4. Find a basis for the span of the following subset of $\mathcal{P}_3$.
$$\{1 + x, 1 - x^2, 3 + 2x - x^2\}$$

5. Give an example to show that the column space of a matrix and the row space of the matrix are, in general, not the same even though they have the same dimension.

6. Suppose that $A \in M_{m\times n}$. Argue that the rank of A is less than or equal to $\min\{m, n\}$.

7. Argue that the rank of a matrix is equal to the rank of its transpose.

8. Describe all matrices that have rank 0. Then find a general description for all matrices of rank 1.

9. Show that the space of all 3-element row vectors (i.e., $M_{1\times 3}$) is isomorphic to $\mathbb{R}^3$.

10. Show that the map $F : \mathcal{P}_2 \to \mathcal{P}_2$ given by $F(ax^2 + bx + c) = bx^2 - (a + c)x + a$ is an isomorphism.

11. For an arbitrary 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A is defined by $\det(A) = ad - bc$. Show that the determinant function is not an isomorphism from $M_{2\times 2}$ into $\mathbb{R}$.

12. Let $B = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$ be a basis for the vector space V. For an arbitrary vector $\vec{v} \in V$, let $\text{Rep}_B(\vec{v})$ be the representation of $\vec{v}$ in terms of B, that is

$$\text{Rep}_B(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + c_3\vec{\beta}_3) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Show that $\text{Rep}_B : V \to \mathbb{R}^3$ is an isomorphism.