Math 236 - Assignment 5

February 21, 2024

Name KEY ____________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due February 28.

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1. Find a basis for the row space, a basis for the column space, and the rank of the matrix A.

$$A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix}$$

   Solution

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 13/11 \\ 0 & 1 & 0 & -17/11 \\ 0 & 0 & 1 & 6/11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Using the RREF, a basis for the row space is

$$B_R = \langle \begin{pmatrix} 1 & 0 & 0 & 13/11 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & -17/11 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 6/11 \end{pmatrix} \rangle,$$

   a basis for the column space is

$$B_C = \left\langle \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ -4 \end{pmatrix} \right\rangle.$$

   and the rank is 3.

2. Consider the matrix $M = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}$.

   (a) Determine whether the row $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is in the row space of M.

   Solution

   The system

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} a + \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} b + \begin{pmatrix} -1 & 2 & 7 \end{pmatrix} c$$

   gives

$$-b - c = 1, \quad a + 2c = 1, \quad 3a + b + 7c = 1.$$

The corresponding augmented matrix and its RREF are:

$$\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 7 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

No, $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is not in the row space of M. The last row of the RREF above shows a contradiction.

(b) Find a basis for the row space of M.

Solution

$$\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the RREF of M, we see that a basis for the row space is

$$B_R = \langle \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} \rangle.$$

(c) Find the representation for the row $\begin{pmatrix} -3 & 8 & 27 \end{pmatrix}$ in terms of your basis.

Solution

The system

$$\begin{pmatrix} -3 & 8 & 27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} a + \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} b$$

gives

$$a = -3, \quad b = 8, \quad -a + 3b = 27.$$

$$\text{Rep}_{B_R}(\begin{pmatrix} -3 & 8 & 27 \end{pmatrix}) = \begin{pmatrix} -3 \\ 8 \end{pmatrix}.$$

3. Find a basis for the span of the following subset of $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix} \right\}$$

Solution

Let's use RREF to find a basis for the row space of the corresponding matrix of rows:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ 1 & -3 & -3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -3/5 \\ 0 & 1 & 4/5 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the span of the vectors has basis

$$B = \left\langle \begin{pmatrix} 1 \\ 0 \\ -3/5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4/5 \end{pmatrix} \right\rangle.$$

4. Find a basis for the span of the following subset of $\mathcal{P}_3$.

$$\{1+x, 1-x^2, 3+2x-x^2\}$$

Solution

Treat the polynomials as rows of a matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 3 & 2 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$B = \langle 1-x^2, x+x^2 \rangle$$

5. Give an example to show that the column space of a matrix and the row space of the matrix are, in general, not the same even though they have the same dimension.

Solution

A very simple example is a 1×2 matrix such as $M = \begin{pmatrix} 1 & 2 \end{pmatrix}$. The row space is the set of all scalar multiples of $\begin{pmatrix} 1 & 2 \end{pmatrix}$, which is a subspace of $\mathcal{M}_{1\times 2}$. On the other hand, the column space is $\mathbb{R}$. Both have dimension 1, but they are distinctly different spaces (but they are isomorphic!).

6. Suppose that $A \in M_{m\times n}$. Argue that the rank of A is less than or equal to $\min\{m, n\}$.

Solution

Since row rank and column rank are always equal, the rank cannot exceed the number of rows (m), and it also cannot exceed the number of columns (n). So $r \leq m$ and $r \leq n$. Therefore $r \leq \min\{m, n\}$.

7. Argue that the rank of a matrix is equal to the rank of its transpose.

Solution

The rows of A^T are the columns of A. Therefore the column rank of A is the row rank of A^T. Similarly, the row rank of A is the column rank of A^T. But since row and column rank are equal, all these ranks have to be the same.

8. Describe all matrices that have rank 0. Then find a general description for all matrices of rank 1.

Solution

The only $m \times n$ rank-zero matrix is the $m \times n$ zero matrix.

A rank-one matrix must have at least one nonzero row (and column). Each nonzero row (or column) must be a scalar multiple of any nonzero row (or column).

9. Show that the space of all 3-element row vectors (i.e., $\mathcal{M}_{1\times 3}$) is isomorphic to $\mathbb{R}^3$.

   Solution

   Define $f : \mathcal{M}_{1\times 3} \to \mathbb{R}^3$ by

   $$f(\begin{pmatrix} x & y & z \end{pmatrix}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   This is obviously an isomorphism. Details omitted.

10. Show that the map $F : \mathcal{P}_2 \to \mathcal{P}_2$ given by $F(ax^2 + bx + c) = bx^2 - (a+c)x + a$ is an isomorphism.

    Solution

    Onto: Let $c_2x^2 + c_1x + c_0$ be an arbitrary element of $\mathcal{P}_2$. Then $c_0x^2 + c_2x + (-c_0 - c_1) \in \mathcal{P}_2$ and

    $$F(c_0x^2 + c_2x + (-c_0 - c_1)) = c_2x^2 + c_1x + c_0.$$

    One-to-one: Suppose $F(c_2x^2 + c_1x + c_0) = F(d_2x^2 + d_1x + d_0)$. Then

    $$c_1x^2 - (c_2 + c_0)x + c_2 = d_1x^2 - (d_2 + d_0)x + d_2.$$

    It follows that

    $$c_1 = d_1 \qquad c_2 + c_0 = d_2 + d_0, \qquad c_2 = d_2,$$

    from which it follows that

    $$c_2 = d_2, \quad c_1 = d_1, \quad c_0 = d_0.$$

    Linear: Let $p(x) = c_2x^2 + c_1x + c_0$ and $q(x) = d_2x^2 + d_1x + d_0$ be arbitrary elements of $\mathcal{P}_2$, and let $\alpha, \beta \in \mathbb{R}$.

    $$F(\alpha p(x) + \beta q(x)) = \cdots = F((\alpha c_2 + \beta d_2)x^2 + (\alpha c_1 + \beta d_1)x + (\alpha c_0 + \beta d_0))$$
    $$= (\alpha c_1 + \beta d_1)x^2 - [\alpha(c_2 + c_0) + \beta(d_2 + d_0)] + (\alpha c_2 + \beta d_2)$$
    $$= \alpha[c_1x^2 - (c_2 + c_0)x + c_2] + \beta[d_1x^2 - (d_2 + d_0)x + d_2]$$
    $$\alpha F(p(x)) + \beta F(q(x)).$$

11. For an arbitrary 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A is defined by $\det(A) = ad - bc$. Show that the determinant function is not an isomorphism from $M_{2\times 2}$ into $\mathbb{R}$.

Solution

The determinant function is not one-to-one. For example, let

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Then $\det(P) = \det(Q) = 0$, but $P \neq Q$.

12. Let $B = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$ be a basis for the vector space V. For an arbitrary vector $\vec{v} \in V$, let $\text{Rep}_B(\vec{v})$ be the representation of $\vec{v}$ in terms of B, that is

$$\text{Rep}_B(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + c_3\vec{\beta}_3) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Show that $\text{Rep}_B : V \to \mathbb{R}^3$ is an isomorphism.

Solution

We proved the general n-dimensional case in class. This is a special case.

Onto: Suppose $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is an arbitrary element of $\mathbb{R}^3$. Then

$$a\vec{\beta}_1 + b\vec{\beta}_2 + c\vec{\beta}_3 \in V$$

and

$$\text{Rep}_B(a\vec{\beta}_1 + b\vec{\beta}_2 + c\vec{\beta}_3) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

One-to-one: Let $\vec{x}, \vec{y} \in V$. The vectors $\vec{x}$ and $\vec{y}$ can be written uniquely in terms of the basis vectors:

$$\vec{x} = x_1\vec{\beta}_1 + x_2\vec{\beta}_2 + x_3\vec{\beta}_3$$

and

$$\vec{y} = y_1\vec{\beta}_1 + y_2\vec{\beta}_2 + y_3\vec{\beta}_3.$$

It follows that

$$\text{Rep}_B(\vec{x}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{and} \qquad \text{Rep}_B(\vec{y}) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Now suppose that $\text{Rep}_B(\vec{x}) = \text{Rep}_B(\vec{y})$. Then we must have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Linear: Let $\vec{x}, \vec{y} \in V$ and $a, b \in \mathbb{R}$. Write $\vec{x}$ and $\vec{y}$ as above:

$$\vec{x} = x_1\vec{\beta}_1 + x_2\vec{\beta}_2 + x_3\vec{\beta}_3$$

and

$$\vec{y} = y_1\vec{\beta}_1 + y_2\vec{\beta}_2 + y_3\vec{\beta}_3.$$

Then

$$a\vec{x} + b\vec{y} = (ax_1 + by_1)\vec{\beta}_1 + (ax_2 + by_2)\vec{\beta}_2 + (ax_3 + by_3)\vec{\beta}_3$$

and

$$\text{Rep}_B(a\vec{x} + b\vec{y}) = \begin{pmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ ax_3 + by_3 \end{pmatrix} = \cdots = a \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + b \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = a\text{Rep}_B(\vec{x}) + b\text{Rep}_B(\vec{y}).$$