Math 236 - Assignment 6

Name ____________________

February 28, 2024

Score __________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due March 6.

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1. Prove that a composition of isomorphisms is an isomorphism. More formally, suppose $g : U \to V$ and $f : V \to W$ are isomorphisms. Show that $f \circ g : U \to W$ is an isomorphism, where $(f \circ g)(x)$ means $f(g(x))$.

2. f and g are functions from $\mathbb{R}^3$ into $\mathbb{R}^2$ as defined below:
$$f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad g(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
Show that one is a homomorphism and one is not.

3. Show that $f : \mathcal{M}_{2\times 2} \to \mathbb{R}$ is a homomorphism.
$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = 2a + 3b + c - d$$

4. Let $d : \mathbb{R}^3 \to \mathbb{R}$ be defined by
$$d(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \cdot (\hat{\imath} + 2\hat{\jmath} - 3\hat{k}).$$
Show that d is a homomorphism. (The centered dot denotes the dot product from Calculus III.)

5. Assume that $h : V \to W$ is a homomorphism. The *null space* of h is
$$\mathcal{N}(h) = \{\vec{v} \in V : h(\vec{v}) = \vec{0}_W\}.$$
Show that the null space is a subspace of V.

6. Assume that $h : V \to W$ is a homomorphism. The *range* of h is
$$\mathcal{R}(h) = \{\vec{w} \in W : \vec{w} = h(\vec{v}) \text{ for some } \vec{v} \in V\}.$$
Show that the range is a subspace of W.

7. For the map $h : \mathbb{R}^3 \to \mathbb{R}^2$ given by
$$h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} x + y \\ x + z \end{pmatrix},$$
find the range space, rank, null space, and nullity.

8. Suppose $h : V \to V$ is a homomorphism and that $B = \langle \vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n \rangle$ is a basis for V. Prove the statement: If $h(\vec{\beta}_i) = \vec{0}$ for each basis vector, then h is the zero map.

9. Make up two different nontrivial homomorphisms from $\mathbb{R}^2$ into $\mathcal{P}_2$. Call them f and g. Prove that $2f + 3g$ is a homomorphism from $\mathbb{R}^2$ into $\mathcal{P}_2$.