Math 236 - Assignment 6

February 28, 2024

Name ____________________

Score __________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due March 6.

1. Prove that a composition of isomorphisms is an isomorphism. More formally, suppose $g : U \to V$ and $f : V \to W$ are isomorphisms. Show that $f \circ g : U \to W$ is an isomorphism, where $(f \circ g)(x)$ means $f(g(x))$.

   Solution

   Onto: Let $\vec{w}$ be an arbitrary vector in W. Since f is onto, there exists a vector $\vec{v} \in V$ such that $f(\vec{v}) = \vec{w}$. Since g is onto, there exists a vector $\vec{u} \in U$ such that $g(\vec{u}) = \vec{v}$. For this $\vec{u}$, we have $f(g(\vec{u})) = f(\vec{v}) = \vec{w}$.

   One-to-one: Suppose $f(g(\vec{u})) = f(g(\vec{v}))$. Since f is ono-to-one, we must have $g(\vec{u}) = g(\vec{v})$, and since g is one-to-one, we must have $\vec{u} = \vec{v}$.

   Linear: $f(g(\alpha\vec{x} + \beta\vec{y})) = f(\alpha g(\vec{x}) + \beta g(\vec{y}))$ since g is a linear map. Furthermore,

   $$f(\alpha g(\vec{x}) + \beta g(\vec{y})) = \alpha f(g(\vec{x})) + \beta f(g(\vec{y}))$$

   since f is a linear map.

2. f and g are functions from $\mathbb{R}^3$ into $\mathbb{R}^2$ as defined below:

   $$f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad g(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   Show that one is a homomorphism and one is not.

   Solution

   f is the zero map. It is a homomorphism:

   $$f(\alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \beta \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha f(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + \beta f(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}).$$

   g is not a homomorphism:

   $$g(\alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \beta \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha g(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + \beta g(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}).$$

3. Show that $f : \mathcal{M}_{2\times 2} \to \mathbb{R}$ is a homomorphism.

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = 2a + 3b + c - d$$

Solution

$$f(\alpha \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \beta \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) = f(\begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{pmatrix})$$

$$= 2(\alpha a_1 + \beta a_2) + 3(\alpha b_1 + \beta b_2) + (\alpha c_1 + \beta c_2) - (\alpha d_1 + \beta d_2)$$

$$= \alpha(2a_1 + 3b_1 + c_1 - d_1) + \beta(2a_2 + 3b_2 + c_2 - d_2) = \alpha f(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + \beta f(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})$$

4. Let $d : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$d(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \cdot (\hat{\imath} + 2\hat{\jmath} - 3\hat{k}).$$

Show that d is a homomorphism. (The centered dot denotes the dot product from Calculus III.)

Solution

First, expand the dot product to write

$$d(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = x + 2y - 3z.$$

Now,

$$d(\alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \beta \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = d(\begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{pmatrix})$$

$$= (\alpha x_1 + \beta x_2) + 2(\alpha y_1 + \beta y_2) - 3(\alpha z_1 + \beta z_2) = \alpha(x_1 + 2y_1 - 3z_1) + \beta(x_2 + 2y_2 - 3z_2)$$

$$= \alpha d(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + \beta d(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}).$$

5. Assume that $h : V \to W$ is a homomorphism. The *null space* of h is

$$\mathcal{N}(h) = \{\vec{v} \in V : h(\vec{v}) = \vec{0}_W\}.$$

Show that the null space is a subspace of V.

Solution

Let α and β be arbitrary scalars, and let $\vec{x}$ and $\vec{y}$ be arbitary vectors in $\mathcal{N}(h)$. It follows that $\alpha h(\vec{x}) + \beta h(\vec{y}) = \vec{0}_W$ because $\vec{x}$ and $\vec{y}$ are in the null space of h. Furthermore, since h is a homomorphism,

$$\vec{0}_W = \alpha h(\vec{x}) + \beta h(\vec{y}) = h(\alpha\vec{x} + \beta\vec{y}).$$

Therefore, $\alpha\vec{x} + \beta\vec{y} \in \mathcal{N}(h)$.

6. Assume that $h : V \to W$ is a homomorphism. The *range* of h is

$$\mathcal{R}(h) = \{\vec{w} \in W : \vec{w} = h(\vec{v}) \text{ for some } \vec{v} \in V\}.$$

Show that the range is a subspace of W.

Solution

Let α and β be arbitrary scalars, and let $\vec{x}$ and $\vec{y}$ be arbitary vectors in $\mathcal{R}(h)$. Because $\vec{x}$ and $\vec{y}$ are in the range space of h, there are vectors $\vec{u}, \vec{v} \in V$ such that

$$h(\vec{u}) = \vec{x} \qquad \text{and} \qquad h(\vec{v}) = \vec{y}.$$

Now since h is a homomorphism, it follows that

$$\alpha\vec{x} + \beta\vec{y} = \alpha h(\vec{u}) + \beta h(\vec{v}) = h(\alpha\vec{u} + \beta\vec{v}).$$

Finally, since V is a vector space, $\alpha\vec{u} + \beta\vec{v} \in V$, and we see that $\alpha\vec{x} + \beta\vec{y}$ is the image under h of a vector in V. That is, $\alpha\vec{x} + \beta\vec{y} \in \mathcal{R}(h)$.

7. For the map $h : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} x + y \\ x + z \end{pmatrix},$$

find the range space, rank, null space, and nullity.

Solution

$$\begin{aligned} \text{Range space} &= \left\{ \begin{pmatrix} x + y \\ x + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ 1 \end{pmatrix} z : x, y, z \in \mathbb{R} \right\} \\ &= \text{span}(\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}) = \mathbb{R}^2 \end{aligned}$$

The rank of h is the dimension of the range space of h. That is 2.

$$\text{Null space} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y = 0 \text{ and } x + z = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ -x \\ -x \end{pmatrix} : x \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} x : x \in \mathbb{R} \right\}$$

$$= \text{span}(\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\})$$

The nullity of h is the dimension of the null space of h. That is 1.

8. Suppose $h : V \to V$ is a homomorphism and that $B = \langle \vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n \rangle$ is a basis for V. Prove the statement: If $h(\vec{\beta}_i) = \vec{0}$ for each basis vector, then h is the zero map.

   Solution

   Let $\vec{v}$ be an arbitrary vector in V. Since B is a basis, there exist constants $c_1, c_2, \ldots, c_n$ such that
   $$\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n.$$
   Now, since h is a homomorphism and h maps each basis vector to $\vec{0}$, we have,
   $$h(\vec{v}) = h(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + c_2 h(\vec{\beta}_2) + \cdots + c_n h(\vec{\beta}_n) = c_1\vec{0} + c_2\vec{0} + \cdots + c_n\vec{0} = \vec{0}.$$
   In summary, we've found that $h(\vec{v}) = \vec{0}$ for any vector in $\vec{v} \in V$.

9. Make up two different nontrivial homomorphisms from $\mathbb{R}^2$ into $\mathcal{P}_2$. Call them f and g. Prove that $2f + 3g$ is a homomorphism from $\mathbb{R}^2$ into $\mathcal{P}_2$.

   Solution

   Let $f : \mathbb{R}^2 \to \mathcal{P}_2$ be defined by
   $$f(\begin{pmatrix} a \\ b \end{pmatrix}) = a + ax + bx^2,$$
   and let $g : \mathbb{R}^2 \to \mathcal{P}_2$ be defined by
   $$g(\begin{pmatrix} a \\ b \end{pmatrix}) = bx + ax^2.$$
   f and g are homomorphisms (details omitted).

   The function $h = 2f + 3g$ has the action
   $$h(\begin{pmatrix} a \\ b \end{pmatrix}) = 2a + (2a + 3b)x + (2b + 3a)x^2.$$
   It is completely routine to show that h is a homomorphism. The details are omitted.