Math 236 - Assignment 7

March 20, 2024

Name ____________________

Score ______________

Show all work to receive full credit. Supply explanations when necessary. This assignment is due March 27.

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1. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ be an arbitrary matrix in $\mathcal{M}_{2\times 3}$. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
$$f(\vec{x}) = A\vec{x}.$$
Prove that f is a homomorphism.

2. Consider the homomorphism $h : \mathbb{R}^2 \to \mathbb{R}^3$ defined by
$$h(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 2x \\ x-y \\ x+3y \end{pmatrix}.$$
Using
$$B = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle \quad \text{and} \quad D = \left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$$
as bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, find $\text{Rep}_{B,D}(h)$.

3. Consider the homomorphism $h : \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ defined by
$$h(ax^2 + bx + c) = \begin{pmatrix} c & b \\ -b & 2a \end{pmatrix}.$$
Using
$$B = \langle 1, x+1, x^2+x+1 \rangle$$
as the basis for $\mathcal{P}_2$ and D as the standard basis for $\mathcal{M}_{2\times 2}$, find $\text{Rep}_{B,D}(h)$.

4. Determine the matrix representing the zero map from $\mathbb{R}^4$ to $\mathbb{R}^2$, with respect to the standard bases.

5. Write the following product as a linear combination of the columns of the matrix.
$$\begin{pmatrix} 2 & 4 & -5 \\ 0 & 8 & 6 \\ -1 & -4 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

6. Write the following product as a linear combination of the rows of the matrix.
$$\begin{pmatrix} 3 & 1 & -6 \end{pmatrix} \begin{pmatrix} 7 & 3 & 2 \\ 1 & 4 & -9 \\ 2 & -3 & 1 \end{pmatrix}$$

7. Make up a 3×3 matrix of rank 3, and call it A. Then make up a 3×3 matrix of rank 2, and call it B. Compute AB and find its rank.

8. A matrix is said to be *upper triangular* if all entries below the main diagonal are zero. For example,

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

   is upper triangular. Argue that the product of two $n \times n$ upper triangular matrices is an upper triangular matrix. (You need not give a formal proof, just a compelling argument.)

9. Find the inverse of $A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.