MTH 236 Assignment 7 Key

1) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$. Let $f(\vec{x}) = A\vec{x}$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

$$f(\alpha\vec{x}) = A\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix} = \begin{pmatrix} a_{11}\alpha x_1 + a_{12}\alpha x_2 + a_{13}\alpha x_3 \\ a_{21}\alpha x_1 + a_{22}\alpha x_2 + a_{23}\alpha x_3 \end{pmatrix}$$

$$= \alpha\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{pmatrix} = \alpha A\vec{x} = \alpha f(\vec{x}) \checkmark$$

$$f(\vec{x}+\vec{y}) = A\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} = \begin{pmatrix} a_{11}(x_1+y_1) + a_{12}(x_2+y_2) + a_{13}(x_3+y_3) \\ a_{21}(x_1+y_1) + a_{22}(x_2+y_2) + a_{23}(x_3+y_3) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{pmatrix} + \begin{pmatrix} a_{11}y_1 + a_{12}y_1 + a_{13}y_3 \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \end{pmatrix}$$

$$= A\vec{x} + A\vec{y} = f(\vec{x}) + f(\vec{y}) \checkmark$$

2) $h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x \\ x-y \\ x+3y \end{pmatrix}$, $B = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle$,

$D = \left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

FIND $\operatorname{Rep}_{B,D}(h)$.

$$\operatorname{Rep}_D\left(h\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \operatorname{Rep}_D\begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} -6 \\ 7.5 \\ 5.5 \end{pmatrix}_D.$$

SOLVE $a\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix}$.

$$\begin{aligned} b - c &= 2 \\ 2a + 2c &= -1 \\ a + b + c &= 7 \end{aligned} \Rightarrow \begin{aligned} a &= -6 \\ b &= 7.5 \\ c &= 5.5 \end{aligned}$$

$$\operatorname{Rep}_D\left(h\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = \operatorname{Rep}_D\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 5 \end{pmatrix}_D$$

SOLVE $a\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$

$$\begin{aligned} b - c &= -2 \\ 2a + 2c &= -2 \\ a + b + c &= 2 \end{aligned} \Rightarrow \begin{aligned} a &= -6 \\ b &= 3 \\ c &= 5 \end{aligned}$$

$$\boxed{\operatorname{Rep}_{B,D}(h) = \begin{pmatrix} -6 & -6 \\ 7.5 & 3 \\ 5.5 & 5 \end{pmatrix}_{B,D}}$$

3) $h(ax^2+bx+c) = \begin{pmatrix} c & b \\ -b & 2a \end{pmatrix}$

$$B = \langle 1, x+1, x^2+x+1 \rangle$$

$$D = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

FIND $\mathrm{Rep}_{B,D}(h)$.

$$\mathrm{Rep}_D(h(1)) = \mathrm{Rep}_D\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_D$$

$$\mathrm{Rep}_D(h(x+1)) = \mathrm{Rep}_D\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}_D$$

$$\mathrm{Rep}_D(h(x^2+x+1)) = \mathrm{Rep}_D\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \end{pmatrix}_D$$

$$\boxed{\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{pmatrix}_{B,D}}$$

4) THE ZERO MAP FROM $\mathbb{R}^4$ TO $\mathbb{R}^2$, WITH RESPECT TO ANY BASES, IS

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5)
$$\begin{pmatrix} 2 & 4 & -5 \\ 0 & 8 & 6 \\ -1 & -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

$$= \boxed{3\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix} + 5\begin{pmatrix} -5 \\ 6 \\ 2 \end{pmatrix}}$$

6)
$$\begin{pmatrix} 3 & 1 & -6 \end{pmatrix}\begin{pmatrix} 7 & 3 & 2 \\ 1 & 4 & -9 \\ 2 & -3 & 1 \end{pmatrix} =$$

$$\boxed{3\begin{pmatrix} 7 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 4 & -9 \end{pmatrix} - 6\begin{pmatrix} 2 & -3 & 1 \end{pmatrix}}$$

7) $A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & -5 \\ 7 & 3 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ So A HAS RANK 3.

$B = \begin{pmatrix} 6 & -3 & 1 \\ 1 & 2 & 3 \\ 3 & -9 & -8 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 11/15 \\ 0 & 1 & 17/15 \\ 0 & 0 & 0 \end{pmatrix}$ B HAS RANK 2.

$AB = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & -5 \\ 7 & 3 & 1 \end{pmatrix}\begin{pmatrix} 6 & -3 & 1 \\ 1 & 2 & 3 \\ 3 & -9 & -8 \end{pmatrix} = \begin{pmatrix} 10 & 5 & 13 \\ 3 & 36 & 43 \\ 48 & -24 & 8 \end{pmatrix}$

$\xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 11/15 \\ 0 & 1 & 17/15 \\ 0 & 0 & 0 \end{pmatrix}$ AB HAS RANK 2.

8) Suppose U and V are $n \times n$ $\underbrace{\text{upper triangular}}^{i,j \text{ entry is zero if } i>j}$ matrices, and let $W = UV$.

Write $U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ where $u_i = i^{\text{th}}$ row of U

and $V = (v_1 \mid v_2 \mid \cdots \mid v_n)$ where $v_j = j^{\text{th}}$ column of V.

It follows that the i,j entry of W is

$$w_{i,j} = u_i \cdot v_j = \begin{pmatrix} 0 \cdots 0 & u_{i,i} & u_{i,i+1} & \cdots & u_{i,n} \end{pmatrix} \begin{pmatrix} v_{1,j} \\ v_{2,j} \\ \vdots \\ v_{j,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now, if $i > j$

that dot product is

$$\underbrace{0 + 0 + \cdots + 0}_{n \text{ times.}} = 0$$

$\therefore$ W is upper triangular.

9) $A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$. FIND A^{-1}!

$$\left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1/2 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1/2 & 1 \\ 0 & 0 & -1 & 1 & 1/2 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & -1 & -1/2 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & -1 & -1/2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & -1 & -1/2 & 1 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1/2 & 1 \\ -1 & -1/2 & 1 \end{pmatrix}}$$