MTH 236 - Assignment 8 key

1.) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

For any θ, $\det(A) = \cos^2\theta + \sin^2\theta = 1 \neq 0$.

$$A^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

2.) $$(A^k)(A^{-1})^k = \underbrace{AA\cdots A}_{k \text{ times}} \cdot \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ times}}$$

$$= I \quad (\text{simplifying from inside out})$$

Also, $$(A^{-1})^k(A^k) = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ times}} \cdot \underbrace{AA\cdots A}_{k \text{ times}}$$

$$= I$$

Therefore $(A^{-1})^k$ is both a left & right inverse of A^k. It follows that $(A^{-1})^k = (A^k)^{-1}$.

3.) Suppose P is a permutation matrix with rows $P_1, P_2, \ldots, P_n$.

Then P^T is the matrix with columns $P_1^T, P_2^T, \ldots, P_n^T$.

Each row P_k and column P_k^T has a single 1 in the same position and zeros elsewhere. No other row or column has a 1 in the same position. It follows that

$P_i \cdot P_j^T = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$ from which it follows that $PP^T = I$.

(Another approach is to recognize that P is an orthogonal matrix.)

4.) Suppose $AB = I$. Then, because $\text{rank}(AB) = n \leq \text{rank}(A)$

and $\text{rank}(AB) \leq \text{rank}(B)$,

A and B must be invertible.

Now,

$$AB = I \iff A^{-1}AB = A^{-1}I \iff B = A^{-1}$$

$$\iff BA = A^{-1}A = I$$

Therefore, we have $AB = I \iff BA = I$.

5.)

Change-of-basis matrix

$$= \left(\text{Rep}_D\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \text{Rep}_D\begin{pmatrix} 2 \\ 2 \end{pmatrix} \right)_{B,D}$$

$0c_1 + 1c_2 = -1$
$4c_1 + 3c_2 = 1$

$c_2 = -1$
$c_1 = 1$

$0c_1 + 1c_2 = 2$
$4c_1 + 3c_2 = 2$

$c_2 = 2$
$c_1 = -1$

$$\boxed{\text{Rep}_D(\text{id}) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}_{B,D}}$$

6.) CHANGE-OF-BASIS MATRIX

$$= \left(\operatorname{Rep}_D(1),\ \operatorname{Rep}_D(x),\ \operatorname{Rep}_D(x^2) \right)_{B,D}$$

$$c_1 x^2 + c_2 + c_3 x = 1. \Rightarrow c_1 = 0,\ c_2 = 1,\ c_3 = 0$$

$$c_1 x^2 + c_2 + c_3 x = x \Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 1$$

$$c_1 x^2 + c_2 + c_3 x = x^2 \Rightarrow c_1 = 1,\ c_2 = 0,\ c_3 = 0$$

$$\boxed{\operatorname{Rep}_D(id) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{B,D}}$$

7.) LET D = BASIS FOR CODOMAIN $= \boxed{\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\rangle.}$

(I JUST MADE UP A BASIS FOR $\mathbb{R}^3$.)

Now $\operatorname{Rep}_D(id) = \left(\operatorname{Rep}_D(\vec{\beta}_1),\ \operatorname{Rep}_D(\vec{\beta}_2),\ \operatorname{Rep}_D(\vec{\beta}_3) \right)_{B,D}$

$$= \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 1 \\ 0 & 1 & 4 \end{pmatrix}_{B,D}$$

$$\operatorname{Rep}_D(\vec{\beta}_1) = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}_D \Rightarrow \vec{\beta}_1 = 3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$\operatorname{Rep}_D(\vec{\beta}_2) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}_D \Rightarrow \vec{\beta}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$\operatorname{Rep}_D(\vec{\beta}_3) = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}_D \Rightarrow \vec{\beta}_3 = 4\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \\ 9 \end{pmatrix}$$

#7 CONTINUED ...

IT FOLLOWS THAT $\boxed{B = \left\langle \begin{pmatrix}3\\2\\5\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix}, \begin{pmatrix}12\\5\\9\end{pmatrix} \right\rangle.}$

8.) $\vec{\beta}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \vec{\beta}_2 = \begin{pmatrix}2\\1\\-3\end{pmatrix}, \quad \vec{\beta}_3 = \begin{pmatrix}3\\3\\3\end{pmatrix}$

$$\vec{K}_1 = \vec{\beta}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\vec{K}_2 = \vec{\beta}_2 - \mathrm{proj}_{\vec{K}_1} \vec{\beta}_2 = \begin{pmatrix}2\\1\\-3\end{pmatrix} + \frac{5}{14}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}33/14\\24/14\\-27/14\end{pmatrix}$$

$$\vec{K}_3 = \vec{\beta}_3 - \mathrm{proj}_{\vec{K}_1} \vec{\beta}_3 - \mathrm{proj}_{\vec{K}_2} \vec{\beta}_3 = \begin{pmatrix}3\\3\\3\end{pmatrix} - \underbrace{\frac{18}{14}}_{\frac{9}{7}}\begin{pmatrix}1\\2\\3\end{pmatrix} - \underbrace{\frac{45/7}{171/14}}_{\frac{10}{19}}\begin{pmatrix}33/14\\24/14\\-27/14\end{pmatrix}$$

$$= \begin{pmatrix}9/19\\-9/19\\3/19\end{pmatrix}$$

$\vec{K}_1 = \begin{pmatrix}1 & 2 & 3\end{pmatrix}^T$

$\vec{K}_2 = \begin{pmatrix}\frac{33}{14} & \frac{24}{14} & \frac{-27}{14}\end{pmatrix}^T$

$\vec{K}_3 = \begin{pmatrix}\frac{9}{19} & \frac{-9}{19} & \frac{3}{19}\end{pmatrix}^T$

OR NORMALIZED

$\vec{u}_1 = \left(\frac{\sqrt{14}}{14}, \frac{\sqrt{14}}{7}, \frac{3\sqrt{14}}{14}\right)^T$

$\vec{u}_2 = \left(\frac{11\alpha}{19}, \frac{8\alpha}{19}, \frac{-9\alpha}{19}\right)^T$

$\vec{u}_3 = \left(\frac{3\sqrt{19}}{19}, \frac{-3\sqrt{19}}{19}, \frac{\sqrt{19}}{19}\right)^T$

WHERE $\alpha = \sqrt{\frac{19}{14}}$.

9.)

$$\begin{aligned} x - y - z + w &= 0 \\ x \quad + z \quad &= 0 \end{aligned} \Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

RREF

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} w$$

$$\text{Basis} = \left\langle \underset{\vec{\beta}_1}{\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}}, \underset{\vec{\beta}_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}} \right\rangle$$

$$\vec{K}_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{K}_2 = \vec{\beta}_2 - \text{proj}_{\vec{K}_1} \vec{\beta}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{(-2)}{6} \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \\ 1 \end{pmatrix}$$

Normalize ...

$$\boxed{\begin{aligned} \frac{\vec{K}_1}{\|\vec{K}_1\|} &= \vec{u}_1 = \left(\frac{-\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, 0 \right)^T \\ \frac{\vec{K}_2}{\|\vec{K}_2\|} &= \frac{\vec{K}_2}{\frac{2\sqrt{3}}{3}} = \left(\frac{-\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2} \right)^T \end{aligned}}$$

10.) IF THE ORIGINAL SET IS DEPENDENT, THE GRAM-SCHMIDT PROCESS WILL EVENTUALLY PRODUCE ZERO VECTORS.

IN FACT, IT WILL PRODUCE A ZERO VECTOR WHENEVER YOU APPLY THE PROCESS TO A VECTOR THAT IS A LINEAR COMBINATION OF PRECEDING VECTORS.

11.) THE DOT PRODUCTS WILL BE ZERO, AND THE PROCESS WILL GENERATE THE ORIGINAL BASIS ITSELF.