MTH 236 - Assignment 9 Key

1) Assume $\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n$ are mutually orthogonal.

Now suppose $c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n = \vec{0}$.

For any $k = 1, 2, \ldots, n$, consider

$$\vec{\beta}_k \cdot \vec{0} = 0 = \vec{\beta}_k \cdot \left(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n\right)$$

$$= \underbrace{0 + 0 + \cdots + 0 + c_k\vec{\beta}_k \cdot \vec{\beta}_k + 0 + \cdots + 0}$$

(using orthogonality: $\vec{\beta}_k \cdot \vec{\beta}_j = 0$ when $k \neq j$)

$$c_k \|\vec{\beta}_k\|^2 = 0 \Rightarrow c_k = 0.$$

This holds for each k from 1 to n.

2) $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \\ 4 & 5 & 1 \end{pmatrix}$

I'll use Gaussian elimination to find $\det(A)$...

$$A \xrightarrow{\substack{R_2 = -2R_1 + R_2 \\ R_3 = -4R_1 + R_3}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 5 & -7 \\ 0 & 9 & -7 \end{pmatrix} \xrightarrow[\text{TIMES } 5]{R_2 = \frac{1}{5}R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -7/5 \\ 0 & 9 & -7 \end{pmatrix} \xrightarrow{R_3 = -9R_1 + R_3} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -7/5 \\ 0 & 0 & 28/5 \end{pmatrix}}_{\det = \frac{28}{5}}$$

$$\boxed{\text{So } \det(A) = 28.}$$

3) $B = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 1 & 1 & -1 & 1 \end{pmatrix}$

$R_2 = -R_1 + R_2$
$R_3 = -R_1 + R_3$

$$B \longrightarrow \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 2 & -2 & -1 \end{pmatrix} \xrightarrow{R_4 = -2R_2 + R_4} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -2 & -3 \end{pmatrix} \xrightarrow{R_4 = R_3 + R_4} \underbrace{\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\det = 2}$$

$\Downarrow$

$\boxed{\det(B) = 2}$

4) $2x - 5y = 7$
$4x + 9y = 4$

$$\Rightarrow \quad x = \frac{\begin{vmatrix} 7 & -5 \\ 4 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 4 & 9 \end{vmatrix}} = \boxed{\frac{83}{38}}$$

$$y = \frac{\begin{vmatrix} 2 & 7 \\ 4 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 4 & 9 \end{vmatrix}} = \frac{-20}{38} = \boxed{-\frac{10}{19}}$$

5) $A = \begin{pmatrix} 12-x & 4 \\ 8 & 8-x \end{pmatrix}$

$$\begin{aligned} \det(A) &= (12-x)(8-x) - 32 \\ &= 96 - 20x + x^2 - 32 \\ &= x^2 - 20x + 64 \\ &= (x-4)(x-16) \end{aligned}$$

A IS SINGULAR IFF $\det(A) = 0$

$\boxed{x = 4, \; x = 16}$

6) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\det(A) = \cos^2\theta + \sin^2\theta = 1$

FOR ANY θ.

$\boxed{A \text{ IS NEVER SINGULAR.}}$

THE COLUMNS ARE NONZERO PERPENDICULAR VECTORS. THEY MUST BE LINEARLY INDEP.

COLUMNS INDEP $\Rightarrow$ $\det \neq 0$.

7) $A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & k & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

$$\det(A) = (-1)^2 \begin{vmatrix} 1 & 1 & -1 \\ 1 & k & 0 \\ 0 & 1 & -1 \end{vmatrix} = (-1)(1) + (-1)(k-1)$$

$$= -k$$

EXPANDED DOWN 2^{ND} COLUMN.

THE SYSTEM HAS A UNIQUE SOLUTION WHENEVER $K \neq 0$.

8) i) A "multiply row k and add to row j" matrix is an identity matrix with a single off diagonal entry replaced with a nonzero number.

Determinant is 1

ii) A "row swap" matrix is identity matrix with two rows swapped.

Determinant is -1.

iii) A "multiply row by number" matrix is the identity matrix with a single diagonal entry replaced by the row multiplier k.

Determinant is k.

9). $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{pmatrix}$

EXPAND DOWN COLUMN 3 ...

$$\det(A) = 0 \cdot A_{13} + (1)\, A_{23} + 0 \cdot A_{33}$$

$$= A_{23} = (-1)^5 \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix}$$

$$= -1(-1-15) = \boxed{16}$$

10) $A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 1 & 8 & 9 \end{pmatrix}$

$$\det(A) = (1)(0-24) - (4)(-9-3) + 3(-8)$$

$$= -24 + 48 - 24 = 0$$

A HAS NO INVERSE,

BUT ITS ADJOINT IS

$$\begin{pmatrix} -24 & -12 & 12 \\ 12 & 6 & -6 \\ -8 & -4 & 4 \end{pmatrix}$$