Math 236 - Test 1
February 7, 2024

Name: Key

Score ________

Show all work to receive full credit. Supply explanations when necessary. Unless otherwise indicated, you may use your calculator to obtain any RREF.

1. (12 points) Reduce to echelon form (not RREF) and then backsolve by substitution. Do not use your calculator for the reduction. Indicate which row operations you used.

$$\begin{aligned} x + y - z &= -2 \\ 2x - 2y + 4z &= 18 \\ x - 2z &= -4 \end{aligned}$$

$R_2 = -2R_1 + R_2$
$R_3 = -R_1 + R_3$ $\Rightarrow$

$$\begin{aligned} x + y - z &= -2 \\ -4y + 6z &= 22 \\ -y - z &= -2 \end{aligned}$$

$R_2 \leftrightarrow R_3 \Rightarrow$

$$\begin{aligned} x + y - z &= -2 \\ -y - z &= -2 \\ -4y + 6z &= 22 \end{aligned}$$

$R_3 = -4R_2 + R_3$

Echelon Form:

$$\begin{aligned} x + y - z &= -2 \\ -y - z &= -2 \\ 10z &= 30 \end{aligned}$$

$z = 3$

$-y = -2 + 3 = 1 \Rightarrow y = -1$

$x = -2 + 3 - (-1) \Rightarrow x = 2$

$\boxed{(x, y, z) = (2, -1, 3)}$

2. (8 points) Explain why the system is not in echelon form. Then reduce it to echelon form (not RREF) and state the leading and free variables. Do not solve the system, and do not use your calculator for the reduction.

$$\begin{aligned} x + 2y - 3z + w &= 8 \\ 6z - 2w &= 8 \\ -3z + w &= -4 \end{aligned}$$

Not echelon form because of the nonzero z-coefficient.

$R_3 = \frac{1}{2}R_2 + R_3$

$\Downarrow$

$$\begin{aligned} x + 2y - 3z + w &= 8 \\ 6z - 2w &= 8 \\ 0 &= 0 \end{aligned}$$

Echelon Form

Leading variables: x, z

Free variables: y, w

3. (9 points) Use your calculator to determine the RREF of the augmented matrix associated with each system. Write the RREF, and then simply say if the system has a unique solution, no solution, or infinitely many solutions.

(a)
$$\begin{array}{rcrcrcrcr} x & & & + & z & + & w & = & 4 \\ 2x & + & y & & & - & w & = & 2 \\ 3x & + & y & + & z & & & = & 7 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \leftarrow \text{INCONSISTENT}$$

$\boxed{\text{NO SOLUTION}}$

(b)
$$\begin{array}{rcrcrcr} 2a & + & b & - & c & = & 2 \\ 2a & & & + & c & = & 3 \\ a & - & b & & & = & 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$\boxed{\text{UNIQUE SOLUTION}}$

IN FACT, IT'S $(a,b,c) = (1,1,1)$

(c)
$$\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 4 \\ x_1 & - & x_2 & + & 2x_3 & = & 5 \\ 4x_1 & - & x_2 & + & 5x_3 & = & 17 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{\text{INFINITELY MANY SOLUTIONS}}$

$\uparrow$ X_3 IS A FREE VARIABLE.

4. (4 points) Say what it means for a matrix to be nonsingular. Then choose from any problem above (problems 1–3) an example of a nonsingular matrix.

A NONSINGULAR MATRIX IS A SQUARE $(n \times n)$ MATRIX THAT IS THE COEFFICIENT MATRIX OF A HOMOGENEOUS SYSTEM WITH A UNIQUE SOLUTION.

(RREF IS THE IDENTITY MATRIX.)

$\underline{\underline{\text{Ex}}}$ FROM 3b
$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

5. (8 points) Write the system as an augmented matrix. Then compute the RREF and determine the complete solution set. You may use your calculator to find the RREF.

$$\begin{array}{rcrcrcrcrcr} 4x_1 & + & 12x_2 & + & x_3 & - & 2x_4 & - & 5x_5 & = & 17 \\ -x_1 & - & 3x_2 & & & + & x_4 & + & x_5 & = & -3 \\ 3x_1 & + & 9x_2 & + & x_3 & - & x_4 & - & 5x_5 & = & 16 \end{array}$$

$$\left(\begin{array}{ccccc|c} 4 & 12 & 1 & -2 & -5 & 17 \\ -1 & -3 & 0 & 1 & 1 & -3 \\ 3 & 9 & 1 & -1 & -5 & 16 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

FREE: x_2, x_4

SOLUTION SET

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_4 \;:\; x_2, x_4 \in \mathbb{R} \right\}$$

6. (9 points) For any integers m and n, we will say $m \sim n$ if and only if $m - n$ is a multiple of 5. Prove that $\sim$ is an equivalence relation.

REFLEXIVE: FOR ANY INTEGER m, $m - m = 0 = 5 \cdot 0$ ✓ $m \sim m$

SYMMETRIC: FOR INTEGERS m & n, ASSUME $m \sim n$ SO THAT $m - n = 5k$, $k \in \mathbb{Z}$

THEN $n - m = 5(-k)$ ✓ $n \sim m$

TRANSITIVE: FOR INTEGERS m, n, k, ASSUME $\underbrace{m - n = 5i}_{m \sim n}$ AND $\underbrace{n - k = 5j}_{n \sim k}$

WHERE $i, j \in \mathbb{Z}$. THEN $m - k = 5i + 5j = 5(i + j)$ ✓

$m \sim k$.

7. (3 points) For real numbers x and y, we will say $x \sim y$ if and only if x is the opposite of y. Prove that $\sim$ is NOT an equivalence relation.

$\sim$ IS NOT REFLEXIVE BECAUSE, IN GENERAL,

A REAL NUMBER IS NOT ITS OWN OPPOSITE.

(IN FACT, ONLY ZERO IS ITS OWN OPPOSITE.)

8. (5 points) List all possible RREFs for a 2×2 matrix.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$a \in \mathbb{R}$

OKAY TO COUNT AS A CASE OF

9. (5 points) Determine whether these matrices are row equivalent. If helpful, you may use your calculator, but explain your reasoning.

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

THEY ARE ROW EQUIVALENT.

TWO MATRICES ARE ROW EQUIV. IFF THEY HAVE THE SAME RREF.

10. (10 points) The following set with the usual addition and scalar multiplication operations in $\mathbb{R}^3$ is a vector space.

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : a + 2b + 3c = 0 \right\}$$

LET $\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$ AND $\alpha \in \mathbb{R}$

Prove the two closure properties and **any one** of the other vector space properties.

① $\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}$ AND SINCE $(a+x) + 2(b+y) + 3(c+z)$

$= (a+2b+3c) + (x+2y+3z) = 0+0$

$= 0$, THE SUM IS IN W. ✓

② $\alpha \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \\ \alpha c \end{pmatrix}$ AND SINCE $\alpha a + 2\alpha b + 3\alpha c$

$= \alpha(a+2b+3c) = \alpha(0) = 0$, THE SCALAR PRODUCT IS IN W. ✓

③ ADDITION IS COMMUTATIVE:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \\ z+c \end{pmatrix}}_{\text{ADDITION IS COMMUTATIVE IN } \mathbb{R}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \checkmark$$

11. (5 points) Show that the following set is a subspace of $M_{2\times 2}$.

$$L = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

IT IS ENOUGH TO SHOW L IS CLOSED UNDER LINEAR COMBOS!

FOR $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in L$

AND $\alpha, \beta \in \mathbb{R}$...

$$\alpha \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + \beta \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} = \begin{pmatrix} \alpha a & 0 \\ \alpha b & \alpha c \end{pmatrix} + \begin{pmatrix} \beta x & 0 \\ \beta y & \beta z \end{pmatrix} = \begin{pmatrix} \alpha a + \beta x & 0 \\ \alpha b + \beta y & \alpha c + \beta z \end{pmatrix} \in L.$$

12. (3 points) Show that the following set is NOT a subspace of $M_{2\times 2}$.

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} : a \in \mathbb{R} \right\}$$

NOT CLOSED UNDER SUMS!

FOR ARBITRARY ELEMENTS OF $\mathcal{U}$, $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$,

WE HAVE $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ a+b & 2 \end{pmatrix}$, WHICH IS NOT IN $\mathcal{U}$.

13. (6 points) Is the vector $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ in the span of this set? You may use your calculator as you see fit, but be sure to show work and/or explain your reasoning.

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} a + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} b = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

NO, THERE IS NO SUCH a, b BECAUSE THE SYSTEM IS INCONSISTENT.

$\Downarrow$

$$\begin{aligned} 2a+b&=1 \\ a-b&=0 \\ -a+b&=3 \end{aligned} \xrightarrow[R_3 = \frac{1}{2}R_1 + R_3]{R_2 = -\frac{1}{2}R_1 + R_2} \begin{aligned} 2a+b&=1 \\ -\tfrac{3}{2}b&=-\tfrac{1}{2} \\ \tfrac{3}{2}b&=\tfrac{7}{2} \end{aligned} \xrightarrow{R_3 = R_2 + R_3} \begin{aligned} 2a+b&=1 \\ -\tfrac{3}{2}b&=-\tfrac{1}{2} \\ 0&=3 \end{aligned}$$

14. (8 points) Consider the subspace of P_3 described below.

$d = a - 2b + c$

$$M = \{a + bx + cx^2 + dx^3 : a - 2b + c - d = 0\}$$

Find a set of vectors whose span is M. Equivalently, find a parameterization for the subspace.

$$M = \{a + bx + cx^2 + (a - 2b + c)x^3 : a, b, c \in \mathbb{R}\}$$

$$= \{a(1+x^3) + b(x - 2x^3) + c(x^2 + x^3) : a, b, c \in \mathbb{R}\}$$

$$= \boxed{\text{span}\{1 + x^3,\ x - 2x^3,\ x^2 + x^3\}}$$

15. (5 points) What does it mean to be a linear combination of x and y? After answering, prove that three linear combinations of x and y is a linear combination of x and y.

A LINEAR COMBINATION OF X & Y IS A SUM OF SCALAR MULTIPLES OF X & Y : $\boxed{\alpha x + \beta y, \text{ WHERE } \alpha, \beta \text{ ARE SCALARS}}$

$$k_1(\alpha_1 x + \beta_1 y) + k_2(\alpha_2 x + \beta_2 y) + k_3(\alpha_3 x + \beta_3 y)$$

$$= k_1\alpha_1 x + k_1\beta_1 y + k_2\alpha_2 x + k_2\beta_2 y + k_3\alpha_3 x + k_3\beta_3 y$$

$$= (k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3)x + (k_1\beta_1 + k_2\beta_2 + k_3\beta_3)y$$

$$= \alpha x + \beta y.$$