Math 236 - Test 2

March 6, 2024

Name ____________________

Score ____________

Show all work to receive full credit. Supply explanations when necessary. You may use your calculator to obtain any RREF.

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1. (8 points) Determine whether the set is a linearly dependent or independent subset of $\mathcal{M}_{2\times 2}$. Then say whether or not it is a basis for $\mathcal{M}_{2\times 2}$.

$$\left\{ \begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 9 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 8 & 1 \\ -11 & 1 \end{pmatrix} \right\}$$

2. (6 points) Explain why the following set in $\mathbb{R}^2$ must be linearly dependent. Then find a two-element linearly independent subset, and prove the linear independence.

$$W = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$$

3. (6 points) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -1 \\ 5 \end{pmatrix}, \qquad \vec{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 4 \\ -1 \end{pmatrix}, \qquad \vec{v}_5 = \begin{pmatrix} 2 \\ 5 \\ 0 \\ 2 \end{pmatrix}$$

4. (3 points) Suppose A is an $n \times n$ matrix. Give three different statements that are equivalent to the statement "A is nonsingular." (The definition does not count as one.)

5. (3 points) Name or describe three different vectors spaces of dimension 6.

6. (6 points) Find a basis for, and the dimension of, the solution set of the following system.

$$\begin{array}{rcrcrcrcl} x_1 & - & 4x_2 & + & 3x_3 & - & x_4 & = & 0 \\ 2x_1 & - & 8x_2 & + & 6x_3 & - & x_4 & = & 0 \end{array}$$

7. (5 points) Determine a basis for the row space of the matrix $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix}$.

   What is the rank of A?

8. (8 points) Consider the function $F : \mathbb{R}^4 \to \mathcal{M}_{2\times 2}$ defined by

$$F(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}) = \begin{pmatrix} c & a+d \\ b & d \end{pmatrix}.$$

Show that F is one-to-one and onto.

9. (3 points) Define three different isomorphisms between $\mathbb{R}^3$ and $\mathcal{P}_2$. You don't need to prove that they are actually isomorphisms (just be sure of it).

10. (5 points) Suppose that $h : V \to W$ is a homomorphism and that $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is a linearly dependent set in V. Prove that $\{h(\vec{v_1}), h(\vec{v_2}), \ldots, h(\vec{v_n})\}$ is a linearly dependent set in W.

11. (6 points) Suppose that $h : V \to V$ is a homomorphism and that $B = \langle \vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n \rangle$ is a basis for V. Prove the statement: If $h(\vec{\beta}_i) = \vec{\beta}_i$ for each basis vector, then h is the identity map (that is, $h(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$).

12. (6 points) Consider $\mathbb{R}^2$ with basis $B = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\rangle$. Suppose $h : \mathbb{R}^2 \to \mathcal{P}_1$ is a homomorphism satisfying

$$h(\begin{pmatrix} 1 \\ 2 \end{pmatrix}) = 3 + 2x \quad \text{and} \quad h(\begin{pmatrix} 0 \\ -1 \end{pmatrix}) = 1 - 4x.$$

Compute $h(\begin{pmatrix} 3 \\ 4 \end{pmatrix})$.

13. (10 points) Consider the homomorphism $h : \mathcal{M}_{2\times 2} \to \mathcal{P}_2$ defined by

$$h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + b + c + dx^2.$$

(a) Before you work any other parts of this problem, determine the sum of the rank of h and the nullity of h, and say how you know.

(b) Find a basis for the range space of h. Then state the rank of h.

(c) Find a basis for the null space of h. Then state the nullity of h.

The following problems are due March 18. You must work on your own.

14. (8 points) A square matrix with a single 1 in each column (or row) and 0's elsewhere is called a *permutation matrix*. For example, P is a 3×3 permutation matrix:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) For the rest of this problem, let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Compute PA and explain the effect of left-multiplying by P.

(b) Compute AP and explain the effect of right-multiplying by P.

(c) What multiplication by what permutation matrix transforms A to the following?

$$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}$$

(d) What multiplication by what permutation matrix transforms A to the following?

$$\begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{pmatrix}$$

15. (5 points) Typically we show that two sets A and B are equal by showing that every element of A is in B and then showing that every element of B is in A.

Let's use this idea to prove that under a homomorphism, the image of a span is the span of the image. In particular, let's prove the following result.

Proposition: Suppose $h : V \to W$ is a homomorphism. Then for any $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ in V, $h\left(\text{span}(\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\})\right) = \text{span}(\{h(\vec{x}_1), h(\vec{x}_2), \ldots, h(\vec{x}_n)\})$.

I'll try to guide you through the proof.

(a) Let $\vec{y}$ be an arbitrary vector in $h\left(\text{span}(\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\})\right)$. Then $\vec{y} = h(\vec{x})$ for some $\vec{x}$ in the span of $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$. Continue this train of thought to show that $\vec{y} \in \text{span}(\{h(\vec{x}_1), h(\vec{x}_2), \ldots, h(\vec{x}_n)\})$.

(b) Now let $\vec{y}$ be an arbitrary vector in $\text{span}(\{h(\vec{x}_1), h(\vec{x}_2), \ldots, h(\vec{x}_n)\})$. Write what this means and continue the train of thought to show that $\vec{y} \in h\left(\text{span}(\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\})\right)$.

The proof of the proposition above is now complete.

16. (12 points) In this problem, we are going to take another look at Gaussian elimination. You may use your calculator or computer to carry out the matrix operations below. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ -1 & -2 & -3 \end{pmatrix}.$$

(a) Let $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and compute $E_1 A$.

(b) Let $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and compute $E_2(E_1 A)$.

(c) Let $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/6 & 1 \end{pmatrix}$ and compute $E_3(E_2 E_1 A)$.

(d) Finally, compute $E_3 E_2 E_1$. Call it L, and notice that L is a *unit lower triangular matrix*. Then compute LA, and notice that $LA = U$, where U is the *upper triangular matrix* you got in part (c).

(e) The matrices E_1, E_2, and E_3 are examples of *elementary matrices*. Multiplication by an elementary matrix performs a single elementary row operation. Look back at the elementary matrices above, and think about how they were chosen to zero out entries in A.

Now let

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & -5 & -21 \\ 1 & -3 & -10 \end{pmatrix}.$$

Find the sequence of elementary matrices that transforms A to an upper triangular matrix. That is, find L so that $LA = U$.